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On Reductive Subgroups of Algebraic Groups and a Question of Külshammer

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Abstract

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This Thesis is motivated by two problems, each concerning representations (homomorphisms) of groups into a connected reductive algebraic group G over an algebraically closed field k. The first problem is due to B. Külshammer and is to do with representations of finite groups in G:

> Let Γ be a finite group and suppose k has characteristic p. Let Γ_p be a Sylow p-subgroup of Γ and let $\rho : \Gamma_p \to G$ be a representation. Are there only finitely many conjugacy classes of representations $\rho' : \Gamma \to G$ whose restriction to Γ_p is conjugate to ρ?

The second problem follows the work of M. Liebeck and G. Seitz: describe the representations of connected reductive algebraic H in G.

These two problems have been settled as long as the characteristic p is large enough but not much is known in the case where the characteristic p is a so called *bad* prime for G, which will be the setting for our work.

At the intersection of these two problems lies another problem which we call the algebraic version of Külshammer's question where we no longer suppose Γ is finite. This new variation of Külshammer's question is interesting in its own right, and a counterexample may provide insight into Külshammer's original question.

Our approach is to convert these problems into problems in the nonabelian 1-cohomology. Let K be a reductive algebraic group, P a parabolic subgroup of G with Levi subgroup $L < P$, V the unipotent radical of P. Let $\rho_0 : K \to L$ be a representation. Then the representations $\rho : K \to P$ that equal ρ_0 under the canonical projection $P \to L$ are in bijective correspondence with elements of the space of 1-cocycles $Z^1(K, V)$ where K acts on V by $x \cdot v = \rho_0(x)v\rho_0(x)^{-1}$. We can then interpret P- and G-conjugacy classes of representations in terms of the 1-cohomology $H^1(K, V)$.

We state and prove the conditions under which a collection of representations from K to P is a finite union of conjugacy classes in terms of the 1-cohomology in Theorem 4.22.

Unlike other approaches, we work directly with the nonabelian 1-cohomology. Even so, we find that the 1-cocycles in $Z^1(K, V)$ often take values in an abelian subgroup of V (Lemmas 5.10 and 5.11). This is interesting, for the question "is the restriction map of 1-cohomologies $H^1(H, V) \to H^1(U, V)$ induced by the inclusion of U in K injective?" is closely linked to the question of Külshammer, and has positive answer if V is abelian and $H = SL_2(k)$ (Example 3.2).

We show that for $G = B_4$ there is a family of pairwise non-conjugate embeddings of SL_2 in G, a direction provided by Stewart who proved the result for $G = F_4$. This is important as an example like this is first needed if one hopes to find a counterexample to the algebraic version of Külshammer's question.

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Contents

To the people of Christchurch affected by the earthquakes of 2010–2011, during which a lot of this work was written.

Chapter 1

Introduction

A major motivation for the work carried out in this thesis is to investigate a question posed by B. Külshammer to do with homomorphisms of finite groups into algebraic groups ([1], [2]). One may call these homomorphisms *representations* because of the obvious similarity with the usual kind of representations into GL_n. Külshammer's second question reads as follows.

> Let G be a linear algebraic group over an algebraically closed field of characteristic p. Let Γ be a finite group and $\Gamma_p < \Gamma$ a Sylow p-subgroup of Γ. Fix a conjugacy class of representations of Γ_p in G. Are there, up to conjugation by G, only finitely many representations $\rho : \Gamma \to G$ whose restrictions to Γ_p belong to the given class?

So far only a non-reductive counterexample is known [1, Appendix]. We examine Külshammer's second question for reductive G.

The work in this thesis also extends the study of the subgroup structure of simple algebraic groups, complementing some of the work done by M. Liebeck and G. Seitz ([3], [4]), and D. Stewart ([5], [6]). Let G be a simple algebraic group over an algebraically closed field of characteristic p. For large enough characteristic ($p = 0$ or $p > 7$ covers all restrictions) Liebeck and Seitz determine explicitly the embeddings of arbitrary connected semisimple subgroups of G, where G is of exceptional type. On the other hand, Stewart's work concerns exceptional groups for the case $p < 7$, so called *low characteristic*. Like Stewart, we examine the subgroup structure of simple algebraic groups in low characteristic (usually $p = 2$ or $p = 3$) where examples are exotic and less is known.

In the intersection of the above two topics of interest lies a variation of Külshammer's second question which we call the *algebraic* version of Külshammer's second question.

In this case we substitute a connected reductive group H for the finite group Γ, instead of a Sylow p-subgroup $\Gamma_p < \Gamma$ we use a maximal unipotent subgroup $U < H$, and by the term representation we mean a homomorphism of algebraic groups. The precise statement of the algebraic version of Külshammer's second question reads as follows.

> Let G, H be connected reductive linear algebraic groups over an algebraically closed field of characteristic p and $U < H$ a maximal unipotent subgroup of H. Fix a conjugacy class of representations of U in G. Are there, up to conjugation by G, only finitely many representations $\rho : H \to G$ whose restrictions to U belong to the given class?

Note that in attempting to answer the algebraic version of Külshammer's second question for H, G we investigate embeddings of H in G, thus building on the work of Liebeck and Seitz, and Stewart.

Furthermore, it may be the case that a counterexample to the algebraic version of Külshammer's second question for some H, G provides a counterexample to the original question for Γ, G where Γ is some finite subgroup of H. For instance, it would be encouraging if there existed a counterexample to the algebraic version of Külshammer's question with $H = SL_2(k), G$, as $SL_2(k)$ has many finite subgroups (e.g. $SL_2(\mathbb{F}_{p^r})$) with which we can test Külshammer's original question.

1.1 G-Complete Reducibility

The notion of G-complete reducibility is due to Serre [7] and extends the notion of completely reducible representations in GL_n to representations in reductive G. Here we state the definition and recall some facts.

A subgroup of G is said to be G-completely reducible if, whenever that subgroup is contained in a parabolic subgroup of G then it is contained in a Levi subgroup of that parabolic. A subgroup of G is said to be G-irreducible if that subgroup is contained in no proper parabolic subgroup of G.

We say that a representation $\rho : H \to G$ is G-completely reducible (respectively, G-irreducible) if its image $\rho(H)$ in G is G-completely reducible (respectively G-irreducible).

If every representation of H in some $GL_n(k)$ is completely reducible we say that H is linearly reductive. In characteristic 0, H is linearly reductive if and only if H is reductive. In characteristic $p > 0$ H is linearly reductive if and only if H° is a torus and H/H° is a finite group of order coprime to p.

If H is a subgroup of G and H is linearly reductive then H is G-completely reducible. Hence G-complete reducibility is uninteresting in characteristic 0.

Let H be a subgroup of G. If L is a Levi subgroup of some parabolic P of G and L contains H, then H is G-completely reducible if and only if H is L-completely reducible (cf. [8, Theorem 3.1]). If H is G-completely reducible then H is L-irreducible for any minimal Levi containing H. More generally, let P be a parabolic containing H, let L be a Levi of P and let $\pi : P \to L$ be the canonical projection. Then P is minimal amongst the parabolics containing H if and only if $\pi(H)$ is L-irreducible.

1.2 Külshammer's Second Question

Külshammer's questions have their roots in Maschke's Theorem, which asserts that any representation from a finite group Γ in $GL_n(k)$ over a field k of characteristic p not dividing the order of Γ is completely reducible. It is a standard result in Representation Theory that there are only finitely many conjugacy classes of completely reducible representations of an arbitrary finite group in $GL_n(k)$. Therefore if $|\Gamma|$ is coprime to p then there are only finitely many conjugacy classes of representations of Γ in $GL_n(k)$.

Külshammer's second question is a refinement of his first question, which reads as follows.

> Suppose p does not divide the order of Γ. Are there only finitely many conjugacy classes of representations of Γ in G?

The answer is positive and the proof is essentially contained in a paper of A. Weil [9]. Külshammer observes this in [2], where he also rephrases his second question.

In the following Example we see infinitely many conjugacy classes of representations of a finite group in $SL_2(k)$.

Example 1.1. *Let $\Gamma = C_p \times C_p = \langle a, b \,|\, ab = ba, a^p = b^p = 1\rangle$ and consider representations $\rho : \Gamma \to SL_2(k)$. In particular, for each $\lambda \in k$ define $\rho_\lambda : \Gamma \to SL_2(k)$ by*

$$\rho_\lambda(a) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},$$
$$\rho_\lambda(b) = \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}.$$

If $\lambda_1 \neq \lambda_2$ then ρ_{λ_1} is not $SL_2(k)$-conjugate to ρ_{λ_2}. Hence there are infinitely many $SL_2(k)$-conjugacy classes of representations of Γ in $SL_2(k)$.

Note that in Külshammer's second question the condition that p does not divide the order of Γ is dropped from the hypothesis.

For $G = GL_n$ the answer is 'yes' (cf. [2, pg. 297]). In [1] Slodowy shows the answer is also 'yes' for reductive G if the prime p is good for G.

More generally, suppose G is reductive and that p does not divide the order of Γ. Then Γ is linearly reductive; that is, every representation from Γ to $GL_n(k)$ is completely reducible. Then by [8, Lemma 2.6] every representation from Γ to G is G-completely reducible. By Theorem 4.3 there are only finitely many G-conjugacy classes of G-completely reducible representations of Γ in G. Therefore the answer to Külshammer's question is 'yes' in this case. On the other hand, if Γ is a p-group then the answer is trivially 'yes'.

In order to find a counterexample for Külshammer's second question for reductive G we need infinitely many non-G-completely reducible representations of Γ. This leads us to study representations into proper parabolics of G.

1.3 Methods

Our approach to Külshammer's second question and to the problem of describing the representations of reductive H in G is to convert the problems into problems involving a certain 1-cohomology of K (= finite Γ or algebraic H), with coefficients in V, the unipotent radical of a parabolic subgroup of G.

This method is inspired by Liebeck and Seitz. The main difference in our calculations is that we deal with the so-called *nonabelian* 1-cohomology directly where as Liebeck, Seitz and Stewart use results from Representation Theory to study *abelian layers* of the 1-cohomology and then piece the layers back together.

Our G_2 calculation in Section 6.3 of Chapter 6 agrees with Stewart's G_2 calculation, and we acknowledge Stewart's F_4 calculation which lead us to find infinitely many conjugacy classes of SL_2 in B_4 (Section 6.5).

1.4 Results and Chapter Overview

One of our main results is Theorem 4.22. With this we are able to relate Külshammer's second question, and the algebraic version, to a certain 1-cohomology calculation in which K (= finite Γ or algebraic H) acts on the unipotent radical V of a parabolic subgroup P of G via a fixed representation $K \to L$ into a Levi subgroup L of P.

We also have a surprising result in Lemma 5.10, where we show that under certain conditions that the 1-cohomology of $SL_2(k), V$ has representative 1-cocycles σ such that $\sigma(B_2(k))$ lies in a product of commuting root groups of V, where $B_2(k)$ is the subgroup of $SL_2(k)$ consisting of upper triangular matrices. This is interesting for the following reason. The algebraic version of Külshammer's second question is closely linked to the question 'is the restriction map of 1-cohomologies $H^1(K,V) \to H^1(U,V)$ injective?'. We show that the answer to the latter question is 'yes' in the case that $K = SL_2(k)$ and V is abelian by reducing to the case $SL_2(\mathbb{F}_q)$ (Example 3.2). Then the hope is that one might be able to reduce the case of nonabelian V to abelian $W < V$, containing the image of σ.

In Chapter 2 we produce some basic facts to do with Linear Algebraic Groups and Root Systems which could be found in texts such as Humphreys [10] and will be well-known to readers with a background in this area. This is an attempt to standardize notation and provide some background for the results to come.

In Chapter 3 we introduce the 1-cohomology, first the well-known abelian case and second the lesser-known nonabelian case. In Example 3.2 we show that the restriction map of 1-cohomologies $H^1(SL_2(k), V) \to H^1(U_2(k), V)$ is injective for $U_2(k)$ the upper unitriangular matrices of $SL_2(k)$ and V a vector space on which $SL_2(k)$ acts linearly. In Lemma 3.27 show that $H^1(SL_2(k), V) \to H^1(B, V)$ is injective for B a Borel subgroup of SL_2 and V an algebraic group, not necessarily abelian, on which SL_2 acts.

Chapter 4 deals with the 1-cohomology in our specific setting of studying Külshammer's second question and studying the subgroup structure of reductive G. We apply the theory and results of the previous Chapter and culminate in Theorem 4.22.

In Chapter 5 we provide some theoretical results for the 1-cohomology calculation for SL_2, V where V is the unipotent radical of a rank 1 parabolic of G. We have evidence that a 1-cocycle in $Z^1(SL_2(k), V)$ under certain conditions has image lying in an abelian subgroup of V (Lemma 5.10). We also simplify the concrete 1-cohomology calculations in the next Chapter by calculating the general form of a 1-cocycle $\sigma : SL_2(k) \to V$ (Lemma 5.11).

In Chapter 6 we calculate the 1-cohomology for $SL_2(k), G$ for G of type B_2. We also provide partial calculations for G of type G_2 and C_3 in order to explore counterexamples to Propositions 5.4 and 5.7 which are related to Lemma 5.10. We also show that there are infinitely many non-G-completely reducible representations of $SL_2(k)$ in G for G of type B_4, which is a necessary condition for a counterexample to the algebraic version of Külshammer's second question.

The future directions of the work in this Thesis are summarized in the final Chapter.

Chapter 2

Mathematical Preliminaries

We will assume the reader has an understanding of basic algebraic geometry and algebraic group theory which could be obtained by consulting Springer [11] or Humphreys [10]. In this Chapter we will recall some relevant parts of the theory of algebraic groups and establish some notation.

Throughout this Thesis, k will denote an algebraically closed field of characteristic $p > 0$, and all varieties and algebraic groups will be defined over k.

Many of our example calculations will involve the special linear group of 2×2 matrices, $SL_2(k)$, and the following subgroups.

$T_2(k)$: the subgroup of diagonal matrices, a maximal torus of $SL_2(k)$,

$B_2(k)$: the subgroup of upper triangular matrices, a Borel subgroup of $SL_2(k)$,

$B_2^-(k)$: the subgroup of lower triangular matrices,

$U_2(k)$: the subgroup of upper unitriangular matrices,

$U_2^-(k)$: the subgroup of lower unitriangular matrices.

$B_2^-(k)$ is the Borel subgroup opposite $B_2(k)$ with respect to $T_2(k)$.

Let T be a maximal torus of a connected reductive linear algebraic group G. We write $\mathfrak{t}$ for the Lie algebra of T. We denote by $X(T)$ the character group of T, defined as the collection of all algebraic group homomorphisms from T to k^* with the addition law

$$(x_1 + x_2)(t) = x_1(t)x_2(t),$$

for all $x_1, x_2 \in X(T), t \in T$.

The cocharacter group, $Y(T)$, is the collection of all algebraic group homomorphisms from k^* to T with the addition law

$$(y_1 + y_2)(\lambda) = y_1(\lambda)y_2(\lambda),$$

for all $y_1, y_2 \in Y(T), \lambda \in k^*$.

If we compose $x \in X(T)$ with $y \in Y(T)$ we get a morphism from k^* to k^*; that is, a morphism of the form $\lambda \mapsto \lambda^n$ for some $n \in \mathbb{Z}$. Hence there exists a pairing $\langle , \rangle : X(T) \times Y(T) \to \mathbb{Z}$ defined

$$(x, y) \mapsto \langle x, y \rangle = n,$$

where $x(y(\lambda)) = \lambda^n$.

Let T be a maximal torus of G. The Lie algebra $\mathfrak{g}$ of G has the decomposition $\mathfrak{g} = \mathfrak{t} \oplus \coprod_{\alpha \in \Phi} \mathfrak{g}_\alpha$, where Φ is the set of roots of G with respect to T. We define the root group U_α to be the unique connected T-stable subgroup of G having Lie algebra $\mathfrak{g}_\alpha$. There exists an isomorphism $\epsilon_\alpha : k \to U_\alpha$ such that for all $t \in T$, $x \in k$, $t\epsilon_\alpha(x)t^{-1} = \epsilon_\alpha(\alpha(t)x)$. The group generated by $\langle U_\alpha, U_{-\alpha} \rangle$ is isomorphic to SL_2 or PGL_2. If U is a connected, T-stable unipotent subgroup of G then $U = \prod_\alpha U_\alpha$, where the product is taken in any fixed order.

If B is a Borel subgroup of G then the positive roots with respect to B will be denoted by Φ^+, the negative roots will be denoted by Φ^-, and the base will be denoted by Δ. Borel subgroups of G containing the maximal torus T are in bijective correspondence with bases for Φ: choosing a Borel of G containing T amounts to choosing a set of simple roots for Φ.

We say that the characteristic p is good for G if p does not divide any of the coefficients of any $\beta \in \Phi$ when β is written as a linear combination of simple roots.

We interpret the roots of G as living in the real vector space $X(T) \otimes \mathbb{R}$ (see the rank 2 root system diagrams in Appendix B.3). Let s_α be the reflection in the hyperplane orthogonal to α. The Weyl Group $W = N_G(T)/T$ is generated by $\{s_\alpha \mid \alpha \in \Delta\}$. If α, β are linearly independent, then there exist $\gamma, \delta \in \Delta$ and $w \in W$ such that $w(\alpha) = \gamma$, while $w(\beta)$ is a $\mathbb{Z}^+$-linear combination of γ, δ.

There is a W-invariant inner product $(\, , \,)$ on $X(T) \otimes \mathbb{R}$; this is uniquely determined up to normalization. We define a function $\langle \, , \, \rangle : X(T) \otimes \mathbb{R} \times X(T) \otimes \mathbb{R} \to \mathbb{R}$ by $\langle \alpha, \beta \rangle = \dfrac{2(\alpha, \beta)}{\|\beta\|}$. For any $\alpha, \beta \in \Phi$, we have $s_\alpha(\beta) = \beta - \langle \beta, \alpha \rangle \alpha$.

Chapter 3

The 1-Cohomology

3.1 Abelian 1-Cohomology

The abelian 1-cohomology is standard (cf. [12]). We present this Section as motivation for the less-known nonabelian theory to appear in the next Section.

If K is an algebraic group and V is an algebraic variety then by an action of K on V we mean an action $K \times V \to V$ which is also a morphism of varieties.

If Γ is a finite group then Γ is a variety over k, and if X is a variety then any function $f : \Gamma \to X$ is a morphism of varieties.

Let K be an algebraic group and V an abelian algebraic group with identity 0 on which K acts by group automorphisms. We denote the action by $x \cdot v$, for $x \in K, v \in V$.

Definition 3.1. We call a morphism $\sigma : K \to V$ a *1-cocycle* if it satisfies

$$\sigma(xy) = \sigma(x) + x \cdot \sigma(y), \tag{3.1}$$

for all x, y in K. Denote by $Z^1(K, V)$ the collection of all 1-cocycles from K to V.

We call Equation 3.1 the *1-cocycle condition.*

For any σ_1, σ_2 in $Z^1(K, V)$,

$$\begin{aligned}
(\sigma_1 + \sigma_2)(xy) &= \sigma_1(xy) + \sigma_2(xy) \\
&= \sigma_1(x) + x \cdot \sigma_1(y) + \sigma_2(x) + x \cdot \sigma_2(y) \\
&= (\sigma_1(x) + \sigma_2(x)) + x \cdot (\sigma_1(y) + \sigma_2(y)) \\
&= (\sigma_1 + \sigma_2)(x) + x \cdot (\sigma_1 + \sigma_2)(y),
\end{aligned}$$

for all $x, y \in K$. Moreover, $\sigma_1 + \sigma_2$ is a morphism, so $Z^1(K, V)$ is closed under pointwise addition.

The trivial map from K to V that sends every $h \in K$ to $0 \in V$ is a 1-cocycle. Define $-\sigma$ by

$$(-\sigma)(x) = -\sigma(x),$$

for all $x \in K$. Then

$$\begin{aligned}(-\sigma)(xy) &= -\sigma(xy) \\ &= -\left(\sigma(x) + x \cdot \sigma(y)\right) \\ &= -\sigma(x) + x \cdot (-\sigma(y)) \\ &= (-\sigma)(x) + x \cdot (-\sigma)(y),\end{aligned}$$

for all $x, y \in K$. Hence $-\sigma \in Z^1(K, V)$. Furthermore, $(\sigma + (-\sigma))(x) = 0$ for all $x \in K$. Therefore each $\sigma \in Z^1(K, V)$ has a negative $-\sigma \in Z^1(K, V)$.

Associativity of pointwise addition in $Z^1(K, V)$ holds because associativity holds in V. Therefore $Z^1\,(K, V)$ is a group. Moreover, since V is abelian, $Z^1(K, V)$ is abelian.

Lemma 3.2. *Let $\sigma \in Z^1(K, V)$. Then $-\sigma(x) = x \cdot \sigma(x^{-1})$ for all $x \in K$.*

Proof. Let $1 \in K$ be the identity. Then

$$\begin{aligned}\sigma(1) \ = \ \sigma(1 \times 1) &= \sigma(1) + 1 \cdot \sigma(1) \\ &= \sigma(1) + \sigma(1) \\ &= 2\,\sigma(1).\end{aligned}$$

This shows $\sigma(1) = 0$. Then, for all $x \in K$,

$$0 = \sigma(1) = \sigma(xx^{-1}) = \sigma(x) + x \cdot \sigma(x^{-1})$$

Hence

$$-\sigma(x) = x \cdot \sigma(x^{-1}),$$

for all $x \in K$. □

Definition 3.3. Let $v \in V$. The morphism $\chi_v^K : K \to V$ defined by

$$\chi_v^K(x) = v - x \cdot v,$$

for all $x \in K$, is called a *1-coboundary.* We denote by $B^1(K,V)$ the collection of all 1-coboundaries from K to V.

For any $v \in V$

$$\begin{aligned}\chi_v^K(xy) &= v - (xy)\cdot v \\ &= v - x\cdot(y\cdot v) \\ &= v - x\cdot(v - v + y\cdot v) \\ &= v - x\cdot v + x\cdot(v - y\cdot v) \\ &= \chi_v^K(x) + x\cdot\chi_v^K(y),\end{aligned}$$

for all $x, y \in K$. Therefore $B^1(K,V) \subset Z^1(K,V)$. Furthermore, for any $v, w \in V$,

$$\begin{aligned}(\chi_v^K - \chi_w^K)(x) &= \chi_v^K(x) - \chi_w^K(x) \\ &= v - x\cdot v - w - x\cdot v \\ &= (v - w) - x\cdot(v - w) \\ &= \chi_{v-w}^K(x),\end{aligned}$$

for all $x \in K$. We see that $B^1(K,V)$ is an abelian subgroup of $Z^1(K,V)$, so we may form the quotient.

Definition 3.4. The *1-cohomology* is the quotient defined by

$$H^1(K,V) = Z^1(K,V)/B^1(K,V).$$

We denote by ψ the canonical projection from $Z^1(K,V)$ to $H^1(K,V)$.

Remark 3.5. We defined the identity of $Z^1(K,V)$ to be the trivial map which sends all $x \in K$ to $0 \in V$. This map is precisely the 1-coboundary χ_0^K. Appropriately, we say a 1-cohomology is *trivial* if $Z^1(K,V) = B^1(K,V)$.

Definition 3.6. Suppose V is a vector space over F. We define scalar multiplication on $Z^1(K,V)$ as follows. For $\lambda \in F, \sigma \in Z^1(K,V)$ define the map $\lambda\sigma$ by $(\lambda\sigma)(x) = \lambda\sigma(x)$ for all $x \in K$.

Lemma 3.7. *If V is a vector space over F and K acts linearly on V then $Z^1(K,V)$ is a vector space and $B^1(K,V) \subset Z^1(K,V)$ is a vector subspace.*

Proof. Let $\lambda \in F, \sigma \in Z^1(K,V)$. Then $\lambda\sigma$ is a 1-cocycle, for

$$\begin{aligned}\lambda\sigma(xy) &= \lambda\left(\sigma(x) + x \cdot \sigma(y)\right) \\ &= \lambda\sigma(x) + \lambda(x \cdot \sigma(y)) \\ &= \lambda\sigma(x) + x \cdot (\lambda\sigma(y)).\end{aligned}$$

So $Z^1(K,V)$ is a subspace of the vector space of functions from K to V. Hence $Z^1(K,V)$ is a vector space.

Let $\chi_v^K \in B^1(K,V)$. Then for all $x \in K$

$$\begin{aligned}\lambda\chi_v^K(x) &= \lambda\left(v - x \cdot v\right) \\ &= \lambda v - \lambda(x \cdot v) \\ &= \lambda v - x \cdot (\lambda v) \\ &= \chi_{\lambda v}^K(x),\end{aligned}$$

Hence $B^1(K,V)$ is a subspace of $Z^1(K,V)$. □

We conclude this Section with a useful Lemma [13, Proposition 1].

Lemma 3.8. *Suppose K is linearly reductive. Then $H^1(K,V)$ is trivial.*

3.2 Nonabelian 1-Cohomology

Richardson introduces the nonabelian 1-cohomology in [14]. We reproduce the definitions and selected results below.

We no longer require that V is abelian, henceforth V is an algebraic group on which K acts by group automorphisms. Accordingly, we denote the identity of V by 1. Much of the preceding Section is a direct analogue; for the most part, formulas are just rewritten in multiplicative notation. One main difference will see is that 1-coboundaries are less useful in the nonabelian setting, especially when defining the 1-cohomology.

Definition 3.9. We call a morphism $\sigma : K \to V$ a *1-cocycle* if it satisfies

$$\sigma(xy) = \sigma(x)(x \cdot \sigma(y)), \tag{3.2}$$

for all $x, y \in K$. Denote by $Z^1(K,V)$ the collection of all 1-cocycles from K to V.

We call Equation 3.2 the *1-cocycle condition.*

Remark 3.10. Unlike the previous Section there is no natural addition operation on $Z^1(K,V)$.

Definition 3.11. Let $v \in V$. The morphism $\chi_v^K : K \to V$ defined by

$$\chi_v^K(x) = v(x \cdot v^{-1}),$$

for all $x \in K$ is called a *1-coboundary.* We denote by $B^1(K,V)$ the collection of all 1-coboundaries from K to V.

For any $v \in V$ and any $x, y \in K$,

$$\begin{aligned}\chi_v^K(xy) &= v\left((xy) \cdot v^{-1}\right)\\ &= v(x \cdot v^{-1})(x \cdot v)((xy) \cdot v^{-1})\\ &= v\left(x \cdot v^{-1}\right)(x \cdot v)\left(x \cdot \left(y \cdot v^{-1}\right)\right)\\ &= v\left(x \cdot v^{-1}\right)\left(x \cdot \left(v\left(y \cdot v^{-1}\right)\right)\right)\\ &= \chi_v^K(x)(x \cdot \chi_v^K(y)),\end{aligned}$$

so $B^1(K,V) \subset Z^1(K,V)$.

In the abelian case we use the fact that we can take the quotient $Z^1(K,V)/B^1(K,V)$, as $B^1(K,V)$ is an abelian subgroup of $Z^1(K,V)$, but in the nonabelian they are just sets. Moreover, although we did not explicitly mention it, in the abelian case the following holds

$$\psi(\sigma_1) = \psi(\sigma_2) \Leftrightarrow \exists v \in V, \sigma_1 = \sigma_2 + \chi_v^K.$$

In the nonabelian case, we have the following.

Lemma 3.12. *Define the relation $\sim$ on $Z^1(K,V)$ by*

$$\sigma_1 \sim \sigma_2 \Leftrightarrow \exists v \in V, \forall x \in K, \sigma_1(x) = v\sigma_2(x)(x \cdot v^{-1}). \tag{3.3}$$

Then $\sim$ is an equivalence relation.

Proof. The relation is symmetric, since $\sigma(x) = 1\sigma(x)(x^{-1} \cdot 1)$ for all $\sigma \in Z^1(K,V)$ and all $x \in K$. It is reflexive since

$$\sigma_1(x) = v\sigma_2(x)(x \cdot v^{-1}) \Rightarrow \sigma_2(x) = v^{-1}\sigma_1(x)(x \cdot v).$$

We show the relation is transitive. Suppose

$$\begin{aligned}\sigma_1(x) &= v\sigma_2(x)(x \cdot v^{-1}), \text{ and}\\ \sigma_2(x) &= w\sigma_3(x)(x \cdot w^{-1}).\end{aligned}$$

Then

$$\begin{aligned}\sigma_1(x) &= v\left(w\sigma_3(x)(x \cdot w^{-1})\right)\left(x \cdot v^{-1}\right)\\ &= vw\sigma_3(x)\left(x \cdot w^{-1}v^{-1}\right)\\ &= (vw)\sigma_3(x)\left(x \cdot (vw)^{-1}\right).\end{aligned}$$

Therefore, Equation 3.3 defines an equivalence relation on $Z^1(K, V)$. □

Definition 3.13. Denote by $H^1(K, V)$ the *1-cohomology*, defined to be the set of equivalence classes of $Z^1(K, V)$ under the relation in Equation 3.3, and denote by ψ the canonical projection from $Z^1(K, V)$ to $H^1(K, V)$.

Remark 3.14. Following Remark 3.5, we identify the trivial map in $Z^1(K, V)$ by χ_1^K, and we say a 1-cohomology is trivial if $H^1(K, V) = \psi(\chi_1^K)$, or equivalently if $Z^1(K, V) = B^1(K, V)$.

R. Richardson [14, Lemma 6.2.6] provides a result analogous to Lemma 3.8.

Lemma 3.15. *Suppose K is linearly reductive and V is unipotent. Then $H^1(K, V)$ is trivial.*

3.3 Maps Between 1-Cohomologies

We use multiplicative notation for the next few definition-like results (Lemma 3.16–Lemma 3.18) but they also hold in the abelian setting as this is just a special case of the nonabelian case. We discuss a consequence of the main Lemma in the abelian setting in Corollary 3.21, and both Lemma 3.25 and Example 3.2 require that V is a vector space. In such cases we revert to additive notation to emphasize the fact that V is abelian.

As in the previous Section, K, V are algebraic groups such that K acts on V by group automorphisms. We denote the action of K on V by $x \cdot v$, for $x \in K, v \in V$.

Lemma 3.16. *Let K' be an algebraic group and let $\zeta : K' \to K$ be a homomorphism. We have an action of K' on V by group automorphisms, defined by $x \cdot v = \zeta(x) \cdot v$ for all $x \in K', v \in V$.*

Then for all $\sigma \in Z^1(K, V)$, $\sigma \circ \zeta \in Z^1(K', V)$.

Proof. Let $\sigma \in Z^1(K, V)$. Evidently $\sigma \circ \zeta$ is a morphism from K' to V. Let $x, y \in K'$, then

$$\begin{aligned}
(\sigma \circ \zeta)(xy) &= \sigma\left(\zeta(xy)\right) \\
&= \sigma\left(\zeta(x)\zeta(y)\right) \\
&= \sigma(\zeta(x))\left(\zeta(x) \cdot \sigma(\zeta(y))\right) \\
&= \sigma(\zeta(x))\left(x \cdot \sigma(\zeta(y))\right) \\
&= (\sigma \circ \zeta)(x)\left(x \cdot (\sigma \circ \zeta)(y)\right).
\end{aligned}$$

Therefore, since $\sigma \circ \zeta$ is a morphism and satisfies the 1-cocycle condition, $\sigma \circ \zeta \in Z^1(K', V)$. □

Lemma 3.17. *Let V' be an algebraic group on which K acts by group automorphisms. Let $\xi : V \to V'$ be a K-equivariant homomorphism, that is, $x \cdot \xi(v) = \xi(x \cdot v)$ for all $x \in K, v \in V$. Then for all $\sigma \in Z^1(K, V)$, $\xi \circ \sigma \in Z^1(K, V')$.*

Proof. Let $\sigma \in Z^1(K, V)$. Evidently $\xi \circ \sigma$ is a morphism from K to V'. Let $x, y \in K$, then

$$\begin{aligned}
(\xi \circ \sigma)(xy) &= \xi\left(\sigma(xy)\right) \\
&= \xi\left(\sigma(x)(x \cdot \sigma(y))\right) \\
&= \xi(\sigma(x))\xi(x \cdot \sigma(y)) \\
&= \xi(\sigma(x))\left(x \cdot \xi(\sigma(y))\right) \\
&= (\xi \circ \sigma)(x)\left(x \cdot (\xi \circ \sigma)(y)\right).
\end{aligned}$$

Therefore, since $\xi \circ \sigma$ is a morphism and satisfies the 1-cocycle condition, $\xi \circ \sigma \in Z^1(K, V')$. □

Lemma 3.18 (Map of 1-Cohomologies)**.** *Let K', V' be algebraic groups such that K' acts on V' by group automorphisms.*

Let $\zeta : K' \to K$ be a homomorphism and let $\xi : V \to V'$ be a K'-equivariant homomorphism; that is, suppose that $\xi(\zeta(x) \cdot v) = x \cdot \xi(v)$ for all $x \in K', v \in V$.

Then the function $Z^1(\zeta, \xi)$ defined by

$$Z^1(\zeta, \xi)(\sigma) = \xi \circ \sigma \circ \zeta,$$

maps $Z^1(K, V)$ to $Z^1(K', V')$.

Furthermore, $Z^1(\zeta,\xi)$ descends to give a well-defined map

$$H^1(\zeta,\xi) : H^1(K,V) \to H^1(K',V'),$$

defined by

$$H^1(\zeta,\xi)(\psi(\sigma)) = \left(\psi' \circ Z^1(\zeta,\xi)\right)(\sigma),$$

for all $\sigma \in Z^1(K,V)$, where ψ' is the canonical projection from $Z^1(K',V')$ to $H^1(K',V')$. Moreover, the following diagram commutes:

$$\begin{tikzcd}
Z^1(K,V) \arrow[r, "Z^1(\zeta,\xi)"] \arrow[d, "\psi"'] & Z^1(K',V') \arrow[d, "\psi'"] \\
H^1(K,V) \arrow[r, "H^1(\zeta,\xi)"] & H^1(K',V').
\end{tikzcd}$$

Proof. Let $\sigma \in Z^1(K,V)$. By Lemma 3.16, $\sigma \circ \zeta \in Z^1(K',V)$, where the action of K' on V is given by

$$x \cdot v = \zeta(x) \cdot v,$$

for $x \in K', v \in V$.

By Lemma 3.17, $\xi \circ (\sigma \circ \zeta) = \xi \circ \sigma \circ \zeta \in Z^1(K',V')$. Therefore $Z^1(\zeta,\xi)$ maps $Z^1(K,V)$ to $Z^1(K',V')$.

It remains to show $H^1(\zeta,\xi)$ is well-defined. Let $\sigma_1,\sigma_2 \in Z^1(K,V)$ such that $\psi(\sigma_1) = \psi(\sigma_2)$. Then there exists $v \in V$ such that

$$\sigma_2(x) = v\sigma_1(x)(x \cdot v^{-1}),$$

for all $x \in K$.

Then, for all $x \in K'$

$$\begin{aligned}
\left(Z^1(\zeta,\xi)(\sigma_2)\right)(x) &= \xi\left(\sigma_2(\zeta(x))\right) \\
&= \xi\left(v\sigma_1(\zeta(x))\left(\zeta(x)\cdot v^{-1}\right)\right) \\
&= \xi(v)\xi(\sigma_1(\zeta(x)))\xi\left(\zeta(x)\cdot v^{-1}\right) \\
&= \xi(v)\xi(\sigma_1(\zeta(x)))\left(x\cdot \xi(v^{-1})\right) \\
&= \xi(v)\left(\left(Z^1(\zeta,\xi)(\sigma_1)\right)(x)\right)\left(x\cdot \xi(v^{-1})\right).
\end{aligned}$$

This shows that $\psi'\left(Z^1(\zeta,\xi)(\sigma_1)\right) = \psi'\left(Z^1(\zeta,\xi)(\sigma_2)\right)$, hence $H^1(\zeta,\xi)$ is well-defined. It is clear that the diagram commutes. $\square$

Remark 3.19. The maps $Z^1(\zeta,\xi), H^1(\zeta,\xi)$ are *functorial*. Suppose that

$$K'' \xrightarrow{\zeta'} K' \xrightarrow{\zeta} K, \text{ and}$$
$$V \xrightarrow{\xi} V' \xrightarrow{\xi'} V'',$$

are homomorphisms of groups, and suppose that K, K', K'' act on V, V', V'' by group automorphisms such that the appropriate equivariance properties hold. Then

$$\begin{aligned} Z^1(\zeta\circ\zeta', \xi'\circ\xi) &= Z^1(\zeta,\xi)\circ Z^1(\zeta',\xi'), \text{ and} \\ H^1(\zeta\circ\zeta', \xi'\circ\xi) &= H^1(\zeta,\xi)\circ H^1(\zeta',\xi') \quad \text{(cf. Equation 4.8)}. \end{aligned}$$

Remark 3.20. The slightly unfortunate choice of notation “$Z^1(K,V)$” does not make the action explicit. A consequence is that given suitable homomorphisms

$$\zeta : K \to K,$$
$$\xi : V \to V,$$

the statement

$$H^1(\zeta,\xi) : H^1(K,V) \to H^1(K,V)$$

is misleading on its own: Is the action of K on V the same on the left and the right? If nothing is said about the action defining the 1-cocycles on the right, we take that to mean the two actions are the same.

Similarly, when ζ is the inclusion of some $K' < K$ in K and ξ is the identity map on V, it is implicit that the action of K' on V is defined by the action of K on V, that is $x\cdot v = \zeta(x)\cdot v$, $x \in K', v \in V$, unless specified otherwise (cf. Example 3.1, Lemma 3.25).

This issue arises in Chapter 4, where we will adopt a modified notation for these 1-cocycles, 1-cohomology, etc., that makes it clear what the action is.

Lemma 3.21. *Let K, K', V, V', ζ, ξ satisfy the requirements of Lemma 3.18 and suppose V, V' are abelian. Then $Z^1(\zeta,\xi)$ is a homomorphism and maps $B^1(K,V)$ into $B^1(K',V')$.*

Moreover, if V, V' are vector spaces and the actions are linear, then $Z^1(K,V)$ is a linear map.

Proof. We prove the case where V, V' are abelian. Let $\sigma_1, \sigma_2 \in Z^1(\zeta, \xi)$. Then

$$\begin{aligned}\left(Z^1(\sigma_1+\sigma_2)\right)(x) &= \xi\left((\sigma_1+\sigma_2)(\zeta(x))\right)\\ &= \xi\left(\sigma_1(\zeta(x))+\sigma_2(\zeta(x))\right)\\ &= \xi\left(\sigma_1(\zeta(x))\right)+\xi\left(\sigma_2(\zeta(x))\right)\\ &= Z^1(\zeta,\xi)(\sigma_1)+Z^1(\zeta,\xi)(\sigma_2).\end{aligned}$$

Clearly $Z^1(\zeta, \xi)\left(\chi_v^K\right) = \chi_{\xi(v)}^{K'}$ for any $v \in V$, so $Z^1(\zeta, \xi)$ maps $B^1(K, V)$ into $B^1(K', V')$.

The case where V, V' are vector spaces is left as an exercise. $\square$

Example 3.1. *Let $K' < K$, let ζ be the inclusion of K' in K, and let ξ be the identity map on V. Then by Lemma 3.18, the map $Z^1(\zeta, \xi)$ defined by*

$$Z^1(\zeta,\xi)(\sigma) = \xi\circ\sigma\circ\zeta = \sigma\circ\zeta,$$

maps $Z^1(K, V)$ into $Z^1(K', V)$, and the map

$$H^1(\zeta,\xi)(\psi(\sigma)) = \psi'\circ Z^1(\zeta,\xi),$$

is a well-defined map of 1-cohomologies from $H^1(K, V)$ to $H^1(K', V)$.

The situation in Example 3.1 will be common, so we introduce further notation to suppress ξ when it is the identity map on V.

Definition 3.22. Let $K' < K$, let ζ be the inclusion of K' in K, and let ξ be the identity map on V. Define

$$\begin{aligned}Z^1(\zeta) &= Z^1(\zeta,\xi),\\ H^1(\zeta) &= H^1(\zeta,\xi).\end{aligned}$$

We call $H^1(\zeta)$ the *restriction of 1-cohomologies.*

Definition 3.23. Denote by $\operatorname{Ker}\left(Z^1(\zeta, \xi)\right)$ the collection of all $\sigma \in Z^1(K, V)$ such that $Z^1(\zeta, \xi)(\sigma) = \chi_1^{K'} \in B^1(K', V')$. Similarly, denote by $\operatorname{Ker}\left(H^1(\zeta, \xi)\right)$ the collection of all $x \in H^1(K, V)$ such that $H^1(\zeta, \xi)(x) = \psi'\left(\chi_1^{K'}\right) \in H^1(K', V')$.

The following Lemma is useful when showing that $H^1(\zeta, \xi)$ is injective (cf. Lemma 3.25, Lemma 3.27).

Lemma 3.24. *Let K, K', V, V', ζ, ξ satisfy the requirements of Lemma 3.18 and suppose ξ is surjective. Let $x \in \operatorname{Ker}\left(H^1(\zeta, \xi)\right)$. Then there exists $\sigma \in Z^1(K, V)$ such that*

$x = \psi(\sigma)$ *and*

$$Z^1(\zeta,\xi)(\sigma) = \chi_1^{K'}.$$

Proof. Let $\tau \in Z^1(K,V)$ such that $\psi(\tau) \in \operatorname{Ker}\left(H^1(\zeta,\xi)\right)$. Hence $Z^1(\zeta,\xi)(\tau)$ is a 1-coboundary, so there exists $v \in V'$ such that

$$Z^1(\zeta,\xi)(\tau) = \chi_v^{K'}.$$

Since ξ is surjective there exists $w \in V$ such that $\xi(w) = v$. Let $\sigma \in Z^1(K,V)$ be defined by

$$\sigma(x) = w^{-1}\tau(x)(x \cdot w),$$

for all $x \in K$. Then $\psi(\tau) = \psi(\sigma)$, and for all $x \in K'$

$$\begin{aligned}
\left(Z^1(\zeta,\xi)(\sigma)\right)(x) &= \xi\left(\sigma(\zeta(x))\right) \\
&= \xi\left(w^{-1}\sigma(\zeta(x))(\zeta(x)\cdot w)\right) \\
&= \xi\left(w^{-1}\right)\xi\left(\sigma(\zeta(x))\right)\xi\left(\zeta(x)\cdot w\right) \\
&= v^{-1}\left(Z^1(\zeta,\xi)(\sigma)\right)(x)(\zeta(x)\cdot v) \\
&= v^{-1}v(\zeta(x)\cdot v^{-1})(\zeta(x)\cdot v) \\
&= (v^{-1}v)\left((\zeta(x)\cdot v)^{-1}(\zeta(x)\cdot v)\right) \\
&= 1.
\end{aligned}$$

Hence $Z^1(\zeta,\xi)(\sigma) = \chi_1^{K'}$. □

The next Lemma is standard. It is a consequence of [12, III.9.5(ii)], the proof of which is left to the reader, so we give our own proof here. The Lemma deals with the abelian 1-cohomology of a finite group, so we alter our notation appropriately.

Lemma 3.25. *Let V be a vector space over k,* $\operatorname{char}(k) = p$*. Let Γ be a finite group that acts linearly on V, and let Γ_p be a* Sylow p-subgroup *of Γ. Let ζ be the inclusion of Γ_p in Γ. Then the map*

$$H^1(\zeta) : H^1(\Gamma,V) \to H^1(\Gamma_p,V)$$

is injective.

Proof. Let $x \in \operatorname{Ker}\left(H^1(\zeta)\right)$. By Lemma 3.24 there exists $\sigma \in Z^1(\Gamma,V)$ such that $x = \psi(\sigma)$ and $\sigma(\gamma) = 0$ for all $\gamma \in \Gamma_p$.

Choose a set of representatives $\{\gamma_1, \dots, \gamma_l\} \subset \Gamma$ for the left coset space of Γ_p in Γ. For any $\gamma \in \Gamma$ and $\gamma' \in \Gamma_p$,

$$\sigma(\gamma\gamma') = \sigma(\gamma) + \gamma \cdot \sigma(\gamma') = \sigma(\gamma) + \gamma \cdot 0 = \sigma(\gamma).$$

Hence σ is constant on the left Γ_p-cosets.

We have an action of Γ on the left Γ_p-coset space, defined by $\gamma \cdot (\widetilde{\gamma}\Gamma_p) = (\gamma\widetilde{\gamma})\Gamma_p$. It is clear that this action is independent of the choice of representative $\widetilde{\gamma}$ of the coset $\widetilde{\gamma}\Gamma_p$, so the action is well-defined. Hence for a fixed $\gamma \in \Gamma$, left multiplication by γ permutes the left Γ_p-cosets $\{\gamma_1\Gamma_p, \gamma_2\Gamma_p, \dots, \gamma_l\Gamma_p\}$. It follow that the ordered set $\{\gamma\gamma_1, \gamma\gamma_2, \dots, \gamma\gamma_l\}$ meets each left Γ_p-coset exactly once. Therefore

$$\sum_{i=1}^{l} \sigma(\gamma\gamma_i) = \sum_{i=1}^{l} \sigma(\gamma_i), \tag{3.4}$$

although the summands may be in a different order.

Let $w = \sum_{i=1}^{l} \sigma(\gamma_i)$ and consider the 1-coboundary $\chi_w^\Gamma \in B^1(\Gamma, V)$.

$$\begin{aligned}
\chi_w^\Gamma(\gamma) &= w - \gamma \cdot w \\
&= \sum_{i=1}^{l} \sigma(\gamma_i) - \gamma \cdot \sum_{i=1}^{l} \sigma(\gamma_i) \\
&= \sum_{i=1}^{l} \sigma(\gamma_i) - \sum_{i=1}^{l} \gamma \cdot \sigma(\gamma_i) \\
&= \sum_{i=1}^{l} \sigma(\gamma_i) - \sum_{i=1}^{l} (\sigma(\gamma\gamma_i) - \sigma(\gamma)) \quad \text{(Equation 3.1)} \\
&= \sum_{i=1}^{l} \sigma(\gamma_i) - \sum_{i=1}^{l} \sigma(\gamma\gamma_i) + \sum_{i=1}^{l} \sigma(\gamma) \\
&= \sum_{i=1}^{l} \sigma(\gamma_i) - \sum_{i=1}^{l} \sigma(\gamma_i) + \sum_{i=1}^{l} \sigma(\gamma) \quad \text{(Equation 3.4)} \\
&= \sum_{i=1}^{l} \sigma(\gamma) \\
&= l\sigma(\gamma),
\end{aligned}$$

for all $\gamma \in \Gamma$.

Since $\gcd(l, p) = \gcd\left([\Gamma_p : \Gamma], p\right) = 1$, the positive integer l is not zero in k. Therefore, by Lemma 3.7

$$\sigma = \chi^{\Gamma}_{l^{-1}w} \in B^1(\Gamma, V),$$

which proves $\mathrm{Ker}\left(H^1(\zeta)\right)$ is trivial. □

Example 3.2. *Let $k = \overline{\mathbb{F}_p} = \bigcup_{r\in\mathbb{N}} \mathbb{F}_{p^r}$. Let V a be vector space over k on which $SL_2(k)$ acts linearly, and let $U_2(k)$ be the subgroup of $SL_2(k)$ consisting of upper unitriangular matrices. Let ζ be the inclusion of $U_2(k)$ in $SL_2(k)$.*

Then the map

$$H^1(\zeta) : H^1(SL_2(k), V) \to H^1(U_2(k), V) \tag{3.5}$$

is injective.

Proof. Let $r \in \mathbb{N}$ and denote the inclusion maps

$$\begin{aligned} \zeta_r &: U_2(\mathbb{F}_{p^r}) \hookrightarrow SL_2(\mathbb{F}_{p^r}),\\ \iota_r &: SL_2(\mathbb{F}_{p^r}) \hookrightarrow SL_2(k),\\ \iota'_r &: U_2(\mathbb{F}_{p^r}) \hookrightarrow U_2(k). \end{aligned}$$

By Lemma 3.18 (Remark 3.19) we get the following commutative diagram,

$$\begin{tikzcd} H^1(SL_2(k), V) \arrow[r, "H^1(\zeta)"] \arrow[d, "H^1(\iota_r)"'] & H^1(U_2(k), V) \arrow[d, "H^1(\iota'_r)"] \\ H^1(SL_2(\mathbb{F}_{p^r}), V) \arrow[r, "H^1(\zeta_r)"'] & H^1(U_2(\mathbb{F}_{p^r}), V). \end{tikzcd} \tag{3.6}$$

It is elementary to show that $U(\mathbb{F}_{p^r})$ is a Sylow p-subgroup of $SL_2(\mathbb{F}_{p^r})$ (Appendix A.1), so by Lemma 3.25, $H^1(\zeta_r)$ is injective for all $r \in \mathbb{N}$.

Let $\sigma \in Z^1(SL_2(k), V)$ such that $\sigma \notin B^1(SL_2(k), V)$, that is,

$$\sigma \neq \chi_v^{SL_2(k)}, \tag{3.7}$$

for any $v \in V$. For each $x \in SL_2(\mathbb{F}_{p^r})$ define the morphism $f_x : V \to V$ by

$$f_x(v) = \sigma(x) - \chi_v^{SL_2(k)}(x).$$

Since $\mathbb{F}_{p^{r!}} \subset \mathbb{F}_{p^{(r+1)!}}$ we have $SL_2(\mathbb{F}_{p^{r!}}) \subset SL_2(\mathbb{F}_{p^{(r+1)!}})$. Consider the sequence $\{C_r\}_{r\in\mathbb{N}}$ defined by

$$C_r = \{v \in V \mid \forall x \in SL_2(\mathbb{F}_{p^r}),\ f_x(v) = 0\}.$$

Then

$$\begin{aligned}\bigcap_{r\in\mathbb{N}} C_{r!} &= \{v \in V \mid \forall x \in SL_2(k),\ f_x(v) = 0\} \\ &= \emptyset \quad \text{(Equation 3.7)}.\end{aligned}$$

Each C_r is closed, and the inclusion $SL_2(\mathbb{F}_{p^{r!}}) \subset SL_2(\mathbb{F}_{p^{(r+1)!}})$ induces the reverse inclusion for the subsequence $C_{r!} \supset C_{(r+1)!}$. Then the Noetherian property for V requires that the subsequence $\{C_{r!}\}_{r\in\mathbb{N}}$ becomes constant, and since $\cap_{r\in\mathbb{N}} C_{r!} = \emptyset$, the subsequence $\{C_{r!}\}_{r\in\mathbb{N}}$ is eventually empty. That is, there exists $s \in \mathbb{N}$ such that

$$Z^1(\iota_s)(\sigma) \neq \chi_v^{SL_2(\mathbb{F}_{p^s})},$$

for any $v \in V$. We have shown that if $\sigma \in Z^1(SL_2(k), V)$ such that $Z^1(\iota_{r!})(\sigma) \in B^1(SL_2(\mathbb{F}_{p^{r!}}), V)$ for all $r \in \mathbb{N}$, then $\sigma \in B^1(SL_2(k), V)$.

So, let $\sigma \in Z^1(SL_2(k), V)$ such that $\psi(\sigma) \in \mathrm{Ker}\left(H^1(\zeta)\right)$. Then, consulting the commutative diagram in Equation 3.6,

$$\begin{aligned}&\psi(\sigma) \in \mathrm{Ker}\left(H^1(\iota'_r) \circ H^1(\zeta)\right), \forall r \in \mathbb{N} \\ \Rightarrow\, &\psi(\sigma) \in \mathrm{Ker}\left(H^1(\zeta_r) \circ H^1(\iota_r)\right), \forall r \in \mathbb{N} \\ \Rightarrow\, &H^1(\iota_r)(\psi(\sigma)) \in \mathrm{Ker}\left(H^1(\zeta_r)\right), \forall r \in \mathbb{N} \\ \Rightarrow\, &H^1(\iota_r)(\psi(\sigma)) \text{ is trivial }, \forall r \in \mathbb{N} \\ \Rightarrow\, &Z^1(\iota_r)(\sigma) \in B^1(SL_2(\mathbb{F}_{p^r}), V), \forall r \in \mathbb{N} \\ \Rightarrow\, &\sigma \in B^1(SL_2(k), V) \\ \Rightarrow\, &\psi(\sigma) \in H^1(SL_2(k), V) \text{ is trivial.}\end{aligned}$$

This shows $H^1(\zeta)$ is injective. □

Remark 3.26. It may be true that the above Example holds for unipotent V. In the Lemma below we show that the restriction map $H^1(SL_2(k), V) \to H^1(B, V)$ is injective, where B is a Borel subgroup of $SL_2(k)$. Then in Lemma 5.10, and Examples 6.3.1 and 6.4.1, we show that for certain G, if $x \in H^1(SL_2(k), V)$ then there exists $\sigma \in Z^1(SL_2(k), V)$ such that $\sigma(B)$ lies in an abelian subgroup of V. The hope is that it is enough that the *image* in V is abelian to be able to show $H^1(B,V) \to H^1(U_2(k),V)$ is injective.

Lemma 3.27. *Let V be an algebraic group, and suppose $SL_2(k)$ acts on V by group automorphisms. Let B be a Borel subgroup of $SL_2(k)$ and let $\zeta : B \to SL_2(k)$ be the inclusion map. Then $H^1(\zeta) : H^1(SL_2(k),V) \to H^1(B,V)$ is injective.*

Proof. Let $x \in \operatorname{Ker}\left(H^1(\zeta)\right)$. Then by Lemma 3.24 there exists $\sigma \in Z^1(SL_2(k),V)$ such that $x = \psi(\sigma)$ and $\sigma(b) = 1$ for all $b \in B$.

Let $y \in SL_2(k), b \in B$. Since $\sigma(yb) = \sigma(y)(y \cdot \sigma(b)) = \sigma(y)$, there exists a unique morphism $\widetilde{\sigma}$ such that the following diagram commutes,

$$\begin{tikzcd} SL_2(k) \arrow[r, "\sigma"] \arrow[d, "\pi"'] & V \\ SL_2/B \arrow[ur, "\widehat{\sigma}"'] & \end{tikzcd} ,$$

where π is the canonical projection $\pi : SL_2 \to SL_2/B$ ([10, §23.3]). Hence $\widehat{\sigma}(xB) = \sigma(x)$ for all $x \in SL_2(k)$.

Since $SL_2(k)/B$ is an irreducible projective variety [10, Theorem 21.3], $\widehat{\sigma}$ must be constant. Therefore, since $\sigma(x) = 1$ for all $x \in B$, $\sigma(x) = \widehat{\sigma}(xB) = 1$ for all $x \in SL_2(k)$. Hence $\sigma = \chi_1^{SL_2(k)} \in B^1(SL_2(k),V)$ and so $\operatorname{Ker}\left(H^1(\zeta)\right)$ is trivial. $\square$

Lemma 3.28. *Suppose V is unipotent and let $x \in H^1(SL_2(k),V)$. Then there exists $\sigma \in Z^1(SL_2(k),V)$ such that $x = \psi(\sigma)$ and $\sigma(T_2(k)) = 1$.*

Proof. Let $\zeta : T_2(k) \to SL_2(k)$ be the inclusion map. Since $T_2(k)$ is linearly reductive, $H^1(T_2(k),V)$ is trivial (Lemma 3.15), so $\operatorname{Ker}\left(H^1(\zeta)\right) = H^1(SL_2(k),V)$. Now apply Lemma 3.24. $\square$

Chapter 4

Külshammer's Second Question

Let K be a linear algebraic group, H a connected reductive algebraic group, and Γ a finite group. As pointed out in the Introduction, Külshammer's second question has positive answer for Γ, G so long as p is good for G. We wish to determine whether there exists a counterexample to Külshammer's second question for some connected reductive G.

Generally speaking, we will be concerned with homomorphisms from K to G. Where possible, we will work in this general setting and state things in terms of K. When we must specifically work with finite K or connected reductive K, we will replace K with Γ or H, respectively.

Definition 4.1. Let X, Y be (algebraic) groups. Then we denote by $\mathrm{Hom}(X, Y)$ the set of (algebraic) group homomorphisms from X to Y.

Remark 4.2. If X is finite then $\mathrm{Hom}(X, Y)$ has the structure of an affine variety over k ([15, Section 3]).

For example, Külshammer's second question as originally stated concerns G-conjugacy classes in $\mathrm{Hom}(\Gamma, G)$ and $\mathrm{Hom}(\Gamma_p, G)$, where Γ_p is a Sylow p-subgroup of Γ.

A consequence of the following Theorem is that if there is a counterexample to Külshammer's second question for some connected reductive G, then since $\mathrm{Hom}(\Gamma, G)$ must contain infinitely many G-conjugacy classes, it must contain infinitely many G-conjugacy classes of non-G-completely reducible homomorphisms.

Theorem 4.3. *There are only finitely many G-conjugacy classes of G-completely reducible homomorphisms from Γ to G.*

Proof. Let $\rho \in \mathrm{Hom}(\Gamma, G)$. Then $G \cdot \rho$ is closed if and only if $\overline{\rho(\Gamma)}$ is strongly reductive ([8, Proposition 2.16]) if and only if $\overline{\rho(\Gamma)}$ is G-completely reducible ([8, Theorem

3.1]). By [16, Theorem 1.2] there are only finitely many closed conjugacy classes of homomorphisms from Γ to G. $\square$

Let G be a connected reductive algebraic group over an algebraically closed field k with $\mathrm{char}(k) = p$. Let P be a parabolic subgroup of G, with Levi subgroup L and unipotent radical V. We have $P = V \rtimes L$, and we denote by π^L the canonical projection

$$\pi^L : P \to L.$$

Since L normalizes V we have an action by group automorphisms of L on V given by

$$l \cdot v = lvl^{-1}, \tag{4.1}$$

for $l \in L, v \in V$.

The following general result allows us to replace G-conjugacy with P-conjugacy.

Lemma 4.4. *Let $R \subset \mathrm{Hom}(K, P)$. Then R is contained in a finite union of G-conjugacy classes if and only if it is contained in a finite union of P-conjugacy classes.*

Proof. Let $\rho_1, \rho_2 \in R$ such that ρ_1 and ρ_2 lie in the same G-conjugacy class of R. Then there exists $g \in G$ such that

$$g\rho_1(x)g^{-1} = \rho_2(x),$$

for all $x \in K$.

Let $Q = gPg^{-1}$, hence $\rho_2(K) \subset P \cap Q$. Let T be a maximal torus of G contained in $P \cap Q$. Since T and gTg^{-1} are maximal tori of Q they must be Q-conjugate, so there exists $q \in Q$ such that

$$qTq^{-1} = gTg^{-1}.$$

Then there exists $r \in P$ such that $q = grg^{-1}$, so

$$grg^{-1}Tgr^{-1}g^{-1} = gTg^{-1}$$
$$\Rightarrow rg^{-1}Tgr^{-1} = T.$$

Therefore $gr^{-1} \in N_G(T)$.

Fix a finite set $N \subset N_G(T)$ of coset representatives for the Weyl group $W = N_G(T)/T$ and let $n \in N, t \in T$ such that

$$gr^{-1} = nt.$$

Let $q' = r^{-1}t^{-1}$ so $q' \in P$. Then

$$\begin{aligned}\rho_1(x) &= g^{-1}\rho_2(x)g \\ &= (q'n^{-1})\rho_2(x)nq'^{-1},\end{aligned}$$

for all $x \in K$. Hence $\rho_1 \in P \cdot (n^{-1} \cdot \rho_2)$. This shows that a G-conjugacy class of R is contained in a union of at most $|N| = |W|$ P-conjugacy classes.

Therefore, if R is contained in a finite union of G-conjugacy classes then it is contained in a finite union of P-conjugacy classes. The converse is trivial. □

4.1 The Application of the 1-Cohomology

Let $\omega \in \mathrm{Hom}(K, L)$. Then via the action of L on V given in Equation 4.1, we have an action of K on V given by

$$x \cdot v = \omega(x) \cdot v, \tag{4.2}$$

for $x \in K, v \in V$.

Definition 4.5. Let $\rho \in \mathrm{Hom}(K, P)$. We associate with ρ the map $\rho^L \in \mathrm{Hom}(K, L)$ defined by $\rho^L = \pi^L \circ \rho$.

Let $\rho \in \mathrm{Hom}(K, P)$ and define the map $\sigma_\rho : K \to V$ by

$$\sigma_\rho(x) = \rho(x)\rho^L(x^{-1}). \tag{4.3}$$

We have an action of K on V via ρ^L using Equation 4.2, and we will show that with this action the map σ_ρ is a 1-cocycle from K to V. To this end, let $x, y \in K$. Then,

$$\begin{aligned}
\sigma_\rho(xy) &= \sigma_\rho(xy)\rho^L(xy)\left(\rho^L(xy)\right)^{-1} \\
&= \rho(xy)\left(\rho^L(xy)\right)^{-1} \\
&= \rho(x)\rho(y)\left(\rho^L(xy)\right)^{-1} \\
&= \sigma_\rho(x)\rho^L(x)\sigma_\rho(y)\rho^L(y)\left(\rho^L(xy)\right)^{-1} \\
&= \sigma_\rho(x)\rho^L(x)\sigma_\rho(y)\rho^L(x^{-1})\rho^L(x)\rho^L(y)\left(\rho^L(xy)\right)^{-1} \\
&= \sigma_\rho(x)\left(x \cdot \sigma_\rho(y)\right).
\end{aligned}$$

Therefore σ_ρ satisfies the 1-cocycle condition in Equation 3.2.

Definition 4.6. Let $\omega \in \mathrm{Hom}(K, L)$. We denote by

$$Z^1(K, V)_\omega$$

the set of 1-cocycles from K to V where K acts on V via ω. Likewise, denote by

$$H^1(K, V)_\omega$$

the 1-cohomology obtained from $Z^1(K, V)_\omega$ under the equivalence relation in Equation 3.3. Denote by ψ the canonical projection

$$\psi : Z^1(K, V)_\omega \to H^1(K, V)_\omega.$$

Definition 4.7. Let $\omega \in \mathrm{Hom}(K, L)$ and define

$$\mathrm{Hom}(K, P)_\omega = \{\rho \in \mathrm{Hom}(K, P) \mid \rho^L = \omega\}.$$

More generally, if $R \subset \mathrm{Hom}(K, P)$ define

$$R_\omega = \{\rho \in R \mid \rho^L = \omega\}.$$

By construction, each $\rho \in \mathrm{Hom}(K, P)_\omega$ yields a 1-cocycle $\sigma_\rho \in Z^1(K, V)_\omega$ using Equation 4.3. Conversely given a 1-cocycle $\sigma \in Z^1(K, V)_\omega$ we can construct a map $\rho : K \to P$ defined by

$$\rho(x) = \sigma(x)\omega(x), \tag{4.4}$$

for all $x \in K$. This construction is a homomorphism from K to P. For take $x, y \in K$, then

$$\begin{aligned}\rho(xy) &= \sigma(xy)\omega(xy)\\ &= \sigma(x)(x \cdot \sigma(y))\omega(x)\omega(y)\\ &= \sigma(x)\omega(x)\sigma(y)\omega(x)^{-1}\omega(x)\omega(y)\\ &= \sigma(x)\omega(x)\sigma(y)\omega(y)\\ &= \rho(x)\rho(y).\end{aligned}$$

We point out that the constructions in Equations 4.3 and 4.4 are inverses of each other, and formalize the preceding discussion in the following Lemma.

Lemma 4.8. *Let $\omega \in \mathrm{Hom}(K, L)$. The map*

$$z : \mathrm{Hom}(K, P)_\omega \to Z^1(K, V)_\omega,$$

defined by

$$z(\rho)(x) = \rho(x)\omega(x^{-1}),$$

for all $\rho \in \mathrm{Hom}(K, P)_\omega$ and all $x \in K$, is a bijection.

Proof. We have previously shown that z is onto and has inverse defined by

$$z^{-1}(\sigma)(x) = \sigma(x)\omega(x),$$

for all $\sigma \in Z^1(K, V)_\omega$ and all $x \in K$, so there is nothing to prove. □

Since $\mathrm{Hom}(K, P)_\omega$ is stable under conjugation by elements of V we can consider V-conjugacy classes of $\mathrm{Hom}(K, P)_\omega$.

Definition 4.9. Denote by $\mathrm{Hom}(K, P)_\omega/V$ the collection of V-conjugacy classes of $\mathrm{Hom}(K, P)_\omega$, and denote by ϕ the canonical projection,

$$\phi : \mathrm{Hom}(K, P)_\omega \to \mathrm{Hom}(K, P)_\omega/V.$$

In fact, we show that z descends to give a bijection from $\mathrm{Hom}(K, P)_\omega/V$ to $H^1(K, V)_\omega$.

Lemma 4.10. *For $\rho \in \mathrm{Hom}(K, P)_\omega$, define*

$$h(\phi(\rho)) = \psi(z(\rho)).$$

Then h is a well-defined bijection from $\mathrm{Hom}(K,P)_\omega/V$ *to* $H^1(K,P)_\omega$. *Moreover, the following diagram commutes:*

$$\begin{tikzcd}
\mathrm{Hom}(K,P)_\omega \arrow[r, "z"] \arrow[d, "\phi"'] & Z^1(K,V)_\omega \arrow[d, "\psi"] \\
\mathrm{Hom}(K,P)_\omega/V \arrow[r, "h"] & H^1(K,V)_\omega.
\end{tikzcd}$$

Proof. Let $\rho_1,\rho_2 \in \mathrm{Hom}(K,P)_\omega$ such that $\phi(\rho_1)=\phi(\rho_2)$. Then there exists $v \in V$ such that

$$\rho_2(x) = v\rho_1(x)v^{-1},$$

for all $x \in K$. Furthermore, for all $x \in K$

$$\begin{aligned}
z(\rho_2)(x) &= \rho_2(x)\omega(x^{-1}) \\
&= v\rho_1(x)v^{-1}\omega(x^{-1}) \\
&= v\rho_1(x)\omega(x^{-1})\omega(x)v^{-1}\omega(x^{-1}) \\
&= v\rho_1(x)\omega(x^{-1})\left(x\cdot v^{-1}\right) \\
&= v\left(z(\rho_1)(x)\right)\left(x\cdot v^{-1}\right).
\end{aligned}$$

This shows that $z(\rho_1)$ and $z(\rho_2)$ satisfy the equivalence relation in Equation 3.3. Therefore $\psi\left(z(\rho_1)\right) = \psi\left(z(\rho_2)\right)$ and so h is well-defined.

Since z and ψ are onto, h is onto. We show h is one-to-one.

Let $\rho_1,\rho_2 \in \mathrm{Hom}(K,P)_\omega$ such that $h(\phi(\rho_1)) = h(\phi(\rho_2))$. Then $\psi(z(\rho_1)) = \psi(z(\rho_2))$. Let $\sigma_1 = z(\rho_1), \sigma_2 = z(\rho_2) \in Z^1(K,V)_\omega$. Then there exists $v \in V$ such that

$$\sigma_2(x) = v\sigma_1(x)\left(x\cdot v^{-1}\right),$$

for all $x \in K$. Then

$$\begin{aligned}
\rho_2(x) &= \sigma_2(x)\omega(x) \\
&= v\sigma_1(x)\left(x\cdot v^{-1}\right)\omega(x) \\
&= v\sigma_1(x)\omega(x)v^{-1}\omega(x^{-1})\omega(x) \\
&= v\sigma_1(x)\omega(x)v^{-1} \\
&= v\rho_1(x)v^{-1} \\
&= (v\cdot\rho_1)(x),
\end{aligned}$$

for all $x \in K$. Hence $\rho_2 = v \cdot \rho_1$ and therefore $\phi(\rho_1) = \phi(\rho_2)$. This shows h is one-to-one. $\square$

Let $q \in P$, and let $v \in V, l \in L$ such that $q = vl$. We can conjugate $\rho \in \mathrm{Hom}(K,P)_\omega$ by q to yield an element of $\mathrm{Hom}(K,P)_{l\cdot\omega}$. For

$$\begin{aligned}\pi^L\left(q\rho(x)q^{-1}\right) &= \pi^L(q)\,\pi^L(\rho(x))\,\pi^L\left(q^{-1}\right)\\ &= l\omega(x)l^{-1},\end{aligned}$$

for all $x \in K$, so $q \cdot \rho \in \mathrm{Hom}(K,P)_{l\cdot\omega}$.

Recall from Definition 4.6 that $Z^1(K,V)_{l\cdot\omega}$ is the collection of 1-cocycles from K to V where K acts on V via $l \cdot \omega$. The formula for the action is then given by

$$x.v = l\omega(x)l^{-1}vl\omega(x^{-1})l^{-1},$$

for all $x \in K, v \in V$. Evidently $Z^1(K,V)_{l\cdot\omega} = z\left(\mathrm{Hom}(K,P)_{l\cdot\omega}\right)$. We show that we can apply Lemma 3.18 to get a map of 1-cohomologies $H^1(K,V)_\omega$ to $H^1(K,V)_{l\cdot\omega}$. To this end, let $K' = K$, $V' = V$ and let $\zeta = \mathrm{id} : K \to K$.

Let $\xi = \xi_l : V \to V'$ be defined by $\xi_l(v) = lvl^{-1}$ for all $v \in V$. Now we show that ξ_l satisfies condition (d) of the Lemma. Let $x \in K, v \in V$. Then

$$\begin{aligned}x.\xi_l(v) &= x.\left(lvl^{-1}\right)\\ &= l\omega(x)l^{-1}\left(lvl^{-1}\right)l\omega(x^{-1})l^{-1}\\ &= l\omega(x)v\omega(x^{-1})l^{-1}\\ &= l\,(x \cdot v)\,l^{-1}\\ &= \xi_l\,(x \cdot v)\,.\end{aligned}$$

Therefore, by Lemma 3.18 the map $Z^1(\zeta,\xi_l) : Z^1(K,V)_\omega \to Z^1(K,V)_{l\cdot\omega}$ defined by

$$Z^1(\zeta,\xi_l)(\sigma) = \xi_l \circ \sigma,$$

for all $\sigma \in Z^1(K,V)_\omega$, descends to give a well-defined map of 1-cohomologies

$$H^1(\zeta,\xi_l) : H^1(K,V)_\omega \to H^1(K,V)_{l\cdot\omega},$$

defined by

$$H^1(\zeta,\xi_l)\left(\psi(\sigma)\right) = \psi'\left(Z^1(\zeta,\xi_l)(\sigma)\right),$$

for all $\sigma \in Z^1(K,V)_\omega$, where ψ' is the canonical projection from $Z^1(K,V)_{l\cdot\omega}$ to $H^1(K,V)_{l\cdot\omega}$.

Evidently, if $l \in C_L(\omega(K))$ and $v \in V$ then $\mathrm{Hom}(K,P)_\omega$ is stable under conjugation by vl and $H^1(\zeta, \xi_l)$ maps $H^1(K,V)_\omega$ into itself.

Lemma 4.11. *We have an action of $C_L(\omega(K))$ on $\mathrm{Hom}(K,V)_\omega/V$ defined by*

$$c \cdot \phi(\rho) = \phi\left(\xi_c \circ \rho\right), \tag{4.5}$$

for $c \in C_L(\omega(K)), \rho \in \mathrm{Hom}(K,V)_\omega$.

Similarly, we have an action of $C_L(\omega(K))$ on $H^1(K,V)_\omega$ defined by

$$c \cdot \psi(\sigma) = H^1(\zeta, \xi_c)\left(\psi(\sigma)\right), \tag{4.6}$$

for $c \in C_L(\omega(K)), \sigma \in Z^1(K,V)_\omega$.

Moreover,

$$h(c \cdot \phi(\rho)) = c \cdot h(\phi(\rho)). \tag{4.7}$$

Remark 4.12. Note that $\xi_l \circ \rho = l \cdot \rho$, where the action on the right is the usual conjugation action of G on $\mathrm{Hom}(K,G)$.

Proof. First we show Equation 4.5 defines a well-defined map. Let $\rho_1, \rho_2 \in \mathrm{Hom}(K,V)_\omega$ such that $\phi(\rho_1) = \phi(\rho_2)$. Then there exists $v \in V$ such that

$$\rho_2(x) = v\rho_1(x)v^{-1},$$

for all $x \in K$. Now let $c \in C_L(\omega(K))$. Since L normalizes V there exists $w \in V$ such that $cv = wc$. Then

$$\begin{aligned} c\rho_2(x)c^{-1} &= cv\rho_1(x)v^{-1}c^{-1} \\ &= wc\rho_1(x)c^{-1}w^{-1}, \end{aligned}$$

for all $x \in K$. Therefore $c \cdot \rho_2 = w \cdot (c \cdot \rho_1)$, that is $\xi_c \circ \rho_2 = w \cdot (\xi_c \circ \rho_1)$. Hence $\phi(\xi_c \circ \rho_2) = \phi(\xi_c \circ \rho_1)$ which shows the map defined by Equation 4.5 is well-defined. Since $\xi_{c_1c_2} = \xi_{c_1} \circ \xi_{c_2}$, it is clear that the group action axioms are satisfied, so Equation 4.5 defines an action of $C_L(\omega(K))$ on $\mathrm{Hom}(K,P)_\omega/V$.

Equation 4.6 is well-defined thanks to Lemma 3.18. If e is the identity element of $C_L(\omega(K))$ then the map $H^1(\zeta, \xi_e) = \mathrm{id} : H^1(K,V)_\omega \to H^1(K,V)_\omega$. Finally, let $c_1, c_2 \in$

$C_L(\omega(K))$. Then

$$\begin{aligned}
H^1(\zeta, \xi_{c_1})\left(H^1(\zeta, \xi_{c_2})(\psi(\sigma))\right) &= H^1(\zeta, \xi_{c_1})\left(\psi(Z^1(\zeta, \xi_{c_2})(\sigma))\right) \\
&= H^1(\zeta, \xi_{c_1})\left(\psi(c_2 \cdot \sigma)\right) \\
&= \psi\left(Z^1(\zeta, \xi_{c_1})(c_2 \cdot \sigma)\right) \\
&= \psi\left(c_1 \cdot (c_2 \cdot \sigma)\right) \\
&= \psi\left(c_1 c_2 \cdot \sigma\right) \\
&= \psi\left(Z^1(\zeta, \xi_{c_1 c_2})(\sigma)\right) \\
&= H^1(\zeta, \xi_{c_1 c_2})\left(\psi(\sigma)\right).
\end{aligned} \tag{4.8}$$

This shows that Equation 4.6 defines an action of $C_L(\omega(K))$ on $H^1(K, V)_\omega$.

To prove Equation 4.7 first notice that $(\xi_c \circ \rho_1)^L = \rho_1^L$, hence

$$\begin{aligned}
(z(\xi_c \circ \rho_1))(x) &= c\rho_1(x)c^{-1}\rho_1^L(x^{-1}) \\
&= c\rho_1(x)\rho_1^L(x^{-1})c^{-1} \\
&= (\xi_c(z(\rho_1)(x))) \\
&= (\xi_c \circ z(\rho_1))(x),
\end{aligned}$$

for all $x \in K$. Therefore $z\left(\xi_c \circ \rho_1\right) = \xi_c \circ z(\rho_1)$, and so

$$\begin{aligned}
h(c \cdot \phi(\rho_1)) &= h(\phi(\xi_c \circ \rho_1)) \\
&= \psi(z(\xi_c \circ \rho_1)) \\
&= \psi(\xi_c \circ z(\rho_1)) \\
&= \psi\left(Z^1(\zeta, \xi_c)(z(\rho_1))\right) \\
&= H^1(\zeta, \xi_c)(\psi(z(\rho_1))) \\
&= c \cdot \psi(z(\rho_1)) \\
&= c \cdot h(\phi(\rho_1)).
\end{aligned}$$

□

Definition 4.13. Denote by $\mathrm{Hom}(K, P)_\omega / VC_L(\omega)$ the collection of $C_L(\omega(K))$-conjugacy classes of $\mathrm{Hom}(K, P)_\omega / V$, equivalently the collection of $VC_L(\omega(K))$-conjugacy classes of $\mathrm{Hom}(K, P)_\omega$, and denote by $\widetilde{\phi}$ the canonical projection,

$$\widetilde{\phi} : \mathrm{Hom}(K, P)_\omega / V \to \mathrm{Hom}(K, P)_\omega / VC_L(\omega).$$

Similarly, denote by $H^1(K,V)_\omega/C_L(\omega)$ the set of orbits of $H^1(K,V)_\omega$ under the action defined in Equation 4.6, and denote by $\widetilde{\psi}$ the canonical projection

$$\widetilde{\psi} : H^1(K,V)_\omega \to H^1(K,V)_\omega/C_L(\omega).$$

For ease of notation, define $\Phi : \mathrm{Hom}(K,P)_\omega \to \mathrm{Hom}(K,P)_\omega/VC_L(\omega)$ by

$$\Phi = \widetilde{\phi} \circ \phi,$$

and define $\Psi : Z^1(K,V)_\omega \to H^1(K,V)_\omega/C_L(\omega)$ by

$$\Psi = \widetilde{\psi} \circ \psi.$$

We have the following Lemma, which shows that the map h descends to give a bijection from $\mathrm{Hom}(K,P)_\omega/VC_L(\omega)$ to $H^1(K,V)_\omega/C_L(\omega)$.

Lemma 4.14. *For $\rho \in \mathrm{Hom}(K,P)_\omega$, define*

$$\widetilde{h}(\Phi(\rho)) = \widetilde{\psi}(h(\phi(\rho))).$$

Then $\widetilde{h}$ is a well-defined bijection from

$$\mathrm{Hom}(K,P)_\omega/VC_L(\omega) \to H^1(K,V)_\omega/C_L(\omega).$$

Moreover, the following diagram commutes:

$$\begin{tikzcd}
\mathrm{Hom}(K,P)_\omega/V \arrow[r, "h"] \arrow[d, "\widetilde{\phi}"'] & H^1(K,V)_\omega \arrow[d, "\widetilde{\psi}"] \\
\mathrm{Hom}(K,P)_\omega/VC_L(\omega) \arrow[r, "\widetilde{h}"] & H^1(K,V)_\omega/C_L(\omega).
\end{tikzcd}$$

Proof. Let $\rho_1, \rho_2 \in \mathrm{Hom}(K,V)_\omega$ such that $\Phi(\rho_1) = \Phi(\rho_2)$. Then there exists $c \in C_L(\omega)$ such that $\phi(\rho_2) = c \cdot \phi(\rho_1)$, and we have

$$\begin{aligned} h(\phi(\rho_2)) &= h(c \cdot \phi(\rho_1)) \\ &= c \cdot h(\phi(\rho_1)) \quad \text{(Lemma 4.11)}. \end{aligned}$$

Hence $\widetilde{\psi}(h(\phi(\rho_1))) = \widetilde{\psi}(h(\phi(\rho_2)))$ and so $\widetilde{h}$ is well-defined.

Now let $\rho_1, \rho_2 \in \mathrm{Hom}(K,P)_\omega$ such that $\widetilde{h}(\Phi(\rho_1)) = \widetilde{h}(\Phi(\rho_2))$. Then $\widetilde{\psi}(h(\phi(\rho_1))) = \widetilde{\psi}(h(\phi(\rho_2)))$, so there exists $c \in C_L(\omega)$ such that $h(\phi(\rho_2)) = c \cdot h(\phi(\rho_1)) = h(c \cdot \phi(\rho_1))$. Since h is bijective, this means $\phi(\rho_2) = c \cdot \phi(\rho_1)$. Therefore $\Phi(\rho_1) = \Phi(\rho_2)$ and so $\widetilde{h}$ is injective. Since h and $\widetilde{\psi}$ are surjective, $\widetilde{h}$ is surjective. Therefore $\widetilde{h}$ is bijective. $\square$

Lemma 4.15. *Let $K' < K$, let ζ the inclusion of K' in K, and let ξ be the identity map on V. Then the map $H^1(\zeta) : H^1(K,V)_\omega \to H^1(K',V)_{\omega\circ\zeta}$ descends to give a well-defined map*

$$\widetilde{H^1}(\zeta) : H^1(K,V)_\omega / C_L(\omega) \to H^1(K',V)_{\omega\circ\zeta}/C_L(\omega\circ\zeta),$$

defined by

$$\widetilde{H^1}(\zeta)(\Psi(\sigma)) = \widetilde{\psi}'\left(H^1(\zeta)(\psi(\sigma))\right),$$

for all $\sigma \in Z^1(K,V)_\omega$. Moreover, the following diagram commutes:

$$\begin{tikzcd}
H^1(K,V)_\omega \arrow[r, "H^1(\zeta)"] \arrow[d, "\widetilde{\psi}"'] & H^1(K',V)_{\omega\circ\zeta} \arrow[d, "\widetilde{\psi}'"] \\
H^1(K,V)_\omega/C_L(\omega) \arrow[r, "\widetilde{H^1}(\zeta)"] & H^1(K',V)_{\omega\circ\zeta}/C_L(\omega\circ\zeta).
\end{tikzcd}$$

Proof. Since $C_L(\omega) < C_L(\omega\circ\zeta)$, $C_L(\omega)$ acts on $H^1(K',V)_{\omega\circ\zeta}$. Let $\sigma_1,\sigma_2 \in Z^1(K,V)_\omega$ such that $\Psi(\sigma_1) = \Psi(\sigma_2)$. Then there exists $c \in C_L(\omega)$ such that $\psi(\sigma_2) = c\cdot\psi(\sigma_1)$. Since $H^1(K,V)_\omega \subset H^1(K',V)_\omega$ and $C_L(\omega) < C_L(\omega\circ\zeta)$, we have

$$H^1(\zeta,\xi)\left(\psi(\sigma_2)\right) = c\cdot H^1(\zeta,\xi)\left(\psi(\sigma_2)\right),$$

and so $\Psi'(\sigma_1) = \Psi'(\sigma_2)$. Therefore $\widetilde{H^1}(\zeta,\xi)$ is well-defined. □

Definition 4.16. We define

$$\mathrm{Hom}(K,P)^L = \{\rho^L \mid \rho \in \mathrm{Hom}(K,P)\}.$$

More generally, when $R \subset \mathrm{Hom}(K,P)$ we define

$$R^L = \{\rho^L \mid \rho \in R\}.$$

Lemma 4.17. *Let $R \subset \mathrm{Hom}(K,P)$. Suppose $R = P\cdot\rho$ for some $\rho \in R$. Then $R^L = L\cdot\rho^L$. More generally, if $R = P\cdot R$ then $R^L = L\cdot R^L$.*

Proof. Let $\rho \in R$. Let $q \in P$, so there exist $v \in V, l \in L$ such that $q = vl$. Since $q \cdot \rho \in R$, and

$$\begin{aligned}(q \cdot \rho)^L(x) &= (q\rho(x)q^{-1})^L \\ &= (vl\rho(x)l^{-1}v^{-1})^L \\ &= \pi^L(v)\pi^L(l)\pi^L(\rho(x))\pi^L\left(l^{-1}\right)\pi^L\left(v^{-1}\right) \\ &= l\rho^L(x)l^{-1},\end{aligned}$$

then $(q \cdot \rho)^L = l \cdot \rho^L \in R^L$. This shows $R^L \subset L \cdot R^L$. Conversely, let $l \in L$. Then $l \cdot \rho \in R$, so $(l \cdot \rho)^L = l \cdot \rho^L \in R^L$. Therefore $L \cdot R^L \subset R^L$ and so $R^L = L \cdot R^L$. □

Lemma 4.18. *Let $R \subset \mathrm{Hom}(K, P)$ and suppose that $R = P \cdot R$. Then for all $\omega \in \mathrm{Hom}(K, L)$, all $\rho \in R_\omega$*

$$R_\omega \cap P \cdot \rho = (VC_L(\omega)) \cdot \rho.$$

Proof. Let $\omega \in \mathrm{Hom}(K, L)$ and choose $\rho \in R_\omega$. Suppose there exists $q \in P$ such that $q \cdot \rho \in R_\omega$, and let $v \in V, l \in L$ such that $q = vl$. Then

$$\begin{aligned}q \cdot \rho \in R_\omega &\Leftrightarrow (vl) \cdot \rho \in R_\omega \\ &\Leftrightarrow ((vl) \cdot \rho)^L = \omega \\ &\Leftrightarrow l \cdot \rho^L = \omega \\ &\Leftrightarrow l \in C_L(\omega).\end{aligned}$$

This shows that $R_\omega \cap P \cdot \rho \subset (VC_L(\omega)) \cdot \rho$. The reverse inclusion follows since $R = P \cdot R$ and R_ω is stable under conjugation by V and $C_L(\omega)$. □

Theorem 4.19. *Let $R \subset \mathrm{Hom}(K, P)$ and suppose $P \cdot R = R$. Then R is a finite union of P-conjugacy classes if and only if*

(i) R^L is a finite union of L-conjugacy classes, and

(ii) for each $\omega \in \mathrm{Hom}(K, L)$, $(\widetilde{h} \circ \Phi)(R_\omega)$ is finite in $H^1(K, V)_\omega / C_L(\omega)$.

Remark 4.20. By Lemma 4.17, if R is a finite union of P-conjugacy classes then R^L is a finite union of L-conjugacy classes. Furthermore, conditions (i) and (ii) are equivalent to

(i′) $R_\omega = \emptyset$ for all but finitely many L-conjugacy classes of $\omega \in \mathrm{Hom}(K, L)$, and

(ii′) for each $\omega \in \mathrm{Hom}(\Gamma, L)$, R_ω is a finite union of $VC_L(\omega)$-conjugacy classes, respectively. We obtain (ii) $\Leftrightarrow$ (ii$'$) by appealing to the bijection $\widetilde{h}$, while (i) $\Leftrightarrow$ (i$'$) is self-evident.

Proof. Suppose R is a finite union of P-conjugacy classes, so there exists a finite set $\mathcal{P} \subset \mathrm{Hom}(\Gamma, P)$ such that

$$R = \bigcup_{\rho \in \mathcal{P}} P \cdot \rho$$

Lemma 4.17 shows that (i) holds. We have

$$\begin{aligned} R_\omega &= R_\omega \cap R \\ &= R_\omega \cap \Big(\bigcup_{\rho \in \mathcal{P}} P \cdot \rho \Big) \\ &= \bigcup_{\rho \in \mathcal{P}} (R_\omega \cap P \cdot \rho)\,. \end{aligned}$$

Then by Lemma 4.18

$$R_\omega = \bigcup_{\rho \in \mathcal{P}} (VC_L(\omega)) \cdot \rho.$$

Hence (ii$'$), and therefore (ii), holds. This proves the forward direction of the Theorem.

Conversely, suppose (i) and (ii) hold and let $\omega \in \mathrm{Hom}(K, L)$. By (ii$'$) there exists a finite set $\mathcal{Q} \subset \mathrm{Hom}(\Gamma, P)$ such that

$$R_\omega = \bigcup_{\rho \in \mathcal{Q}} (VC_L(\omega)) \cdot \rho.$$

Applying Lemma 4.18, we get

$$\begin{aligned} R_\omega &= \bigcup_{\rho \in \mathcal{Q}} (R_\omega \cap P \cdot \rho) \\ &= R_\omega \cap \Big(\bigcup_{\rho \in \mathcal{Q}} P \cdot \rho \Big). \end{aligned}$$

Hence R_ω is contained in a finite union of P-conjugacy classes. Define

$$L \cdot R_\omega = \{L \cdot \rho \,|\, \rho \in R_\omega\},$$

so $L \cdot R_\omega$ is contained in a finite union of P-conjugacy classes.

Now let $\rho \in R$. By (i) there exists a finite set $\mathcal{S} \subset \mathrm{Hom}(\Gamma, L)$ such that

$$R^L = \bigcup_{\tau \in \mathcal{S}} L \cdot \tau.$$

Then there exists $l \in L$, $\tau \in \mathcal{S}$ such that $\rho^L = l \cdot \tau$. Hence $l^{-1} \cdot \rho^L = \tau$, so $\rho \in L \cdot R_\tau$. Since R_τ is contained in a finite union of P-conjugacy classes, this shows that R is contained in a finite union of P-conjugacy classes. Therefore, since $R = P \cdot R$, R is a finite union of P-conjugacy classes. $\square$

Definition 4.21. Let $K' < K$ and let ζ be the inclusion of K' in K. Define the maps

$$\begin{aligned}
\mathcal{Z}(\zeta) :&\mathrm{Hom}(K, P)_\omega \to \mathrm{Hom}(K', P)_{\omega\circ\zeta},\\
\mathcal{H}(\zeta) :&\mathrm{Hom}(K, P)_\omega/V \to \mathrm{Hom}(K', P)_{\omega\circ\zeta}/V,\\
\widetilde{\mathcal{H}}(\zeta) :&\mathrm{Hom}(K, P)_\omega/VC_L(\omega) \to \mathrm{Hom}(K', P)_{\omega\circ\zeta}/VC_L(\omega\circ\zeta),
\end{aligned}$$

by

$$\begin{aligned}
\mathcal{Z}(\zeta)(\rho) &= \rho \circ \zeta,\\
\mathcal{H}(\zeta)(\phi(\rho)) &= \phi'(\rho \circ \zeta),\\
\widetilde{\mathcal{H}}(\zeta)(\Phi(\rho)) &= \Phi'(\rho \circ \zeta).
\end{aligned}$$

It is clear that the maps $\mathcal{H}(\zeta), \widetilde{\mathcal{H}}(\zeta)$ are well-defined.

The situation thus far is summarized by the following commutative diagram.

$$
\begin{tikzcd}
\mathrm{Hom}(K,P)_\omega \arrow[r, "z"] \arrow[d, "\phi"] \arrow[dddr, "\mathcal{Z}(\zeta)"] & Z^1(K,V)_\omega \arrow[d, "\psi"] \arrow[dddr, "Z^1(\zeta,\xi)"] & \\
\mathrm{Hom}(K,P)_\omega/V \arrow[r, "h"] \arrow[d, "\widetilde{\phi}"] \arrow[dddr, "\mathcal{H}(\zeta)"] & H^1(K,P)_\omega \arrow[d, "\widetilde{\psi}"] \arrow[dddr, "H^1(\zeta,\xi)"] & \\
\mathrm{Hom}(K,P)_\omega/VC_L(\omega) \arrow[r, "\widetilde{h}"] \arrow[dddr, "\widetilde{\mathcal{H}}(\zeta)"] & H^1(K,P)_\omega/C_L(\omega) \arrow[dddr, "\widetilde{H^1}(\zeta,\xi)"] & \\
& \mathrm{Hom}(K',P)_{\omega\circ\zeta} \arrow[r, "z'"] \arrow[d, "\phi'"] & Z^1(K',V)_\omega \arrow[d, "\psi'"] \\
& \mathrm{Hom}(K',P)_{\omega\circ\zeta}/V \arrow[r, "h'"] \arrow[d, "\widetilde{\phi'}"] & H^1(K',V)_{\omega\circ\zeta} \arrow[d, "\widetilde{\psi'}"] \\
& \mathrm{Hom}(K',P)_{\omega\circ\zeta}/VC_L(\omega\circ\zeta) \arrow[r, "\widetilde{h'}"] & H^1(K',V)_{\omega\circ\zeta}/C_L(\omega\circ\zeta)
\end{tikzcd}
$$

We are now ready to state the main Theorem of this Chapter.

Theorem 4.22. *Let $K' < K$, let ζ be the inclusion of K' in K, and let ξ be the identity map on V. Let $R \subset \mathrm{Hom}(K,P)$ such that $R = P \cdot R$, and let $S = \mathcal{Z}(\zeta)(R)$. Suppose*

(i) R^L is a finite union of L-conjugacy classes,

(ii) for all $\omega \in \mathrm{Hom}(K,L)$ such that $R_\omega \neq \emptyset$, the map

$$\widetilde{H^1}(\zeta,\xi) : H^1(K,V)_\omega/C_L(\omega) \to H^1(K',V)_{\omega\circ\zeta}/C_L(\omega\circ\zeta),$$

has finite fibres, and

(iii) S is a finite union of P-conjugacy classes.

Then R is a finite union of P-conjugacy classes.

Remark 4.23. Since $R = P\cdot R$, then by Lemma 4.17 R^L is already a union of L-conjugacy classes. The point of (i) is that the union is finite.

Proof. Since $P \cdot R = R$, it follows that $P \cdot S = S$. By definition $S \subset \mathrm{Hom}(K',V)$ and by (iii) S is a finite union of P-conjugacy classes. Therefore by Theorem 4.19

(iv) S^L is a finite union of L-conjugacy classes, and

(v) for each $\omega \in \mathrm{Hom}(K', L)$, $(\widetilde{h'} \circ \Phi')\,(S_\omega)$ is finite in $H^1(K', V)_{\omega\circ\zeta}/C_L(\omega \circ \zeta)$.

Let $\omega \in \mathrm{Hom}(K, L)$. Clearly $(\widetilde{h} \circ \Phi)\,(R_\omega)$ is finite if $R_\omega = \emptyset$, so suppose $R_\omega \neq \emptyset$.

We have $\mathcal{Z}(\zeta)(R_\omega) \subset S_{\omega\circ\zeta}$, and by the commutative diagram in Definition 4.21,

$$\widetilde{h'} \circ \Phi' \circ \mathcal{Z}(\zeta) = \widetilde{H^1}(\zeta) \circ \widetilde{h} \circ \Phi.$$

Therefore

$$\begin{aligned}
\left(\widetilde{H^1}(\zeta) \circ \widetilde{h} \circ \Phi\right)(R_\omega) &= \left(\widetilde{h'} \circ \Phi' \circ \mathcal{Z}(\zeta)\right)(R_\omega) \\
&= (\widetilde{h'} \circ \Phi')\,(\mathcal{Z}(\zeta)(R_\omega)) \\
&\subset (\widetilde{h'} \circ \Phi')(S_{\omega\circ\zeta}).
\end{aligned}$$

Then $\left(\widetilde{h'} \circ \Phi'\right)(S_{\omega\circ\zeta})$ is finite by (v), and

$$(\widetilde{h} \circ \Phi)(R_\omega) \subset \widetilde{H^1}(\zeta)^{-1}\left(\left(\widetilde{H^1}(\zeta) \circ \widetilde{h} \circ \Phi\right)(R_\omega)\right),$$

is finite by (ii).

Therefore $(\widetilde{h} \circ \Phi)(R_\omega)$ is finite in any case. Together with (i) we may apply Theorem 4.19, so R is a finite union of P-conjugacy classes. □

Remark 4.24. It is straightforward to show that $H^1(K, V)_\omega/C_L(\omega)$ is finite if and only if $H^1(K, V)_w/C_L(\omega)^\circ$ is finite.

4.2 A Non-Reductive Counterexample

In [1, Appendix], Cram shows a counterexample to Külshammer's second question using a non-reductive algebraic group over an algebraically closed field of characteristic $p = 2$.

Example 4.1. (Cram [1, Appendix]) *Let Q be the algebraic group isomorphic to the affine space $\mathbf{A}^3$ with the group multiplication law:*

$$\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} \times \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} u_1 + v_1 \\ u_2 + v_2 \\ u_3 + v_3 + u_1 v_1 + u_2 v_2 + u_1 v_2 \end{pmatrix}.$$

Let $F = \langle \omega, \tau | \omega^3 = \tau^2 = 1, \tau\omega\tau = \omega^2 \rangle = S_3$ and $F_2 = \langle \tau \rangle$, a Sylow 2-subgroup of F. We have an action of F on Q given by

$$\tau \cdot \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} u_2 \\ u_1 \\ u_3 + u_1^2 + u_2^2 + u_1u_2 \end{pmatrix}$$

$$\omega \cdot \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} u_2 \\ u_1 + u_2 \\ u_3 \end{pmatrix}.$$

Let $G = Q \rtimes F$ and fix the representation $\rho : F_2 \to G$ defined by the natural inclusion $F_2 \to F \to G$. Then there are infinitely many distinct G-conjugacy classes of extensions of ρ to representations of F in G.

Motivated by Theorem 4.22, we examine the counterexample via the 1-cohomology. Note however that the results in this Chapter are stated for reductive G, so they cannot be applied directly.

Choose a 1-cocycle $\sigma \in Z^1(F, Q)$ such that $\sigma|_{\langle \omega \rangle} = 1$. Let

$$\sigma(\tau) = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix},$$

for some $u_1, u_2, u_3 \in k$. Since τ is an involution we have

$$\begin{aligned} 1 = \sigma(\tau^2) &= \sigma(\tau) \times \tau \cdot \sigma(\tau) \\ &= \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} \times \begin{pmatrix} u_2 \\ u_1 \\ u_3 + u_1^2 + u_2^2 + u_1u_2 \end{pmatrix} \\ &= \begin{pmatrix} u_1 + u_2 \\ u_1 + u_2 \\ 2u_3 + 2u_1^2 + u_2^2 + 3u_1u_2 \end{pmatrix} \\ &= \begin{pmatrix} u_1 + u_2 \\ u_1 + u_2 \\ u_2^2 + u_1u_2 \end{pmatrix}. \end{aligned}$$

This shows $u_1 = u_2$, so

$$\sigma(\tau) = \begin{pmatrix} u_1 \\ u_1 \\ u_3 \end{pmatrix}.$$

Furthermore, as $\tau\omega\tau = \omega^2$ we obtain

$$\begin{aligned} 1 = \sigma(\omega^2) &= \sigma(\tau\omega\tau) \\ &= \sigma(\tau) \times \tau \cdot \sigma(\omega\tau) \\ &= \sigma(\tau) \times \tau \cdot \sigma(\omega) \times \tau\omega \cdot \sigma(\tau) \\ &= \sigma(\tau) \times \tau\omega \cdot \sigma(\tau) \\ &= \begin{pmatrix} u_1 \\ u_1 \\ u_3 \end{pmatrix} \times \tau\omega \cdot \begin{pmatrix} u_1 \\ u_1 \\ u_3 \end{pmatrix} \\ &= \begin{pmatrix} u_1 \\ u_1 \\ u_3 \end{pmatrix} \times \tau \cdot \begin{pmatrix} u_1 \\ 0 \\ u_3 \end{pmatrix} \\ &= \begin{pmatrix} u_1 \\ u_1 \\ u_3 \end{pmatrix} \times \begin{pmatrix} 0 \\ u_1 \\ u_3 + u_1^2 \end{pmatrix} \\ &= \begin{pmatrix} u_1 \\ 0 \\ 2u_3 + 3u_1^2 \end{pmatrix} \\ &= \begin{pmatrix} u_1 \\ 0 \\ u_1^2 \end{pmatrix}. \end{aligned}$$

Therefore $u_1 = 0$. Hence a typical 1-cocycle that is trivial on $\langle\omega\rangle$ satisfies

$$\sigma_u(\tau) = \begin{pmatrix} 0 \\ 0 \\ u \end{pmatrix}, \qquad (u \in k).$$

Now we calculate the class $\psi(\sigma_u) \in H^1(F, Q)$. Suppose σ_v, σ_u lie in the same equivalence class. Then there is a $q \in Q$ fixed under the action of ω, that is of the form

$$q = \begin{pmatrix} 0 \\ 0 \\ \lambda \end{pmatrix},$$

such that $\sigma_v(\gamma) = q \times \sigma_u(\gamma) \times \gamma \cdot q^{-1}$. In particular, for $\gamma = \tau$

$$\begin{aligned} \begin{pmatrix} 0 \\ 0 \\ v \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \\ \lambda \end{pmatrix} \times \begin{pmatrix} 0 \\ 0 \\ u \end{pmatrix} \times \tau \cdot \begin{pmatrix} 0 \\ 0 \\ \lambda \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ 0 \\ \lambda \end{pmatrix} \times \begin{pmatrix} 0 \\ 0 \\ u \end{pmatrix} \times \begin{pmatrix} 0 \\ 0 \\ \lambda \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ 0 \\ \lambda \end{pmatrix} \times \begin{pmatrix} 0 \\ 0 \\ u + \lambda \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ 0 \\ u \end{pmatrix}. \end{aligned}$$

Hence only if $u = v$ are two 1-cocycles of the particular form in the same class, and therefore $H^1(F, Q)$ is infinite.

It is natural to ask whether Example 4.1 leads to a reductive counterexample to Külshammer's second question, although we can quickly verify that the answer is "not immediately". For suppose there was a reductive group with unipotent radical having the multiplication law

$$\begin{aligned} &\dots \epsilon_\sigma(u_\sigma) \dots \epsilon_\beta(u_\beta) \dots \epsilon_\gamma(u_\gamma) \times \dots \epsilon_\sigma(v_\sigma) \dots \epsilon_\beta(v_\beta) \dots \epsilon_\gamma(v_\gamma) \\ &= \dots \epsilon_\sigma(u_\sigma + v_\sigma) \dots \epsilon_\beta(u_\beta + v_\beta) \dots \epsilon_\gamma(u_\gamma + v_\gamma + u_\sigma v_\sigma + u_\beta v_\beta + u_\sigma v_\beta). \end{aligned}$$

Then setting $u_\delta = v_\delta = 0$ whenever $\delta \neq \sigma$ gives

$$\epsilon_\sigma(u_\sigma) \times \epsilon_\sigma(v_\sigma) = \epsilon_\sigma(u_\sigma + v_\sigma)\epsilon_\gamma(u_\sigma v_\sigma),$$

which is absurd.

Chapter 5

1-Cohomology Calculations: Theoretical Results

In this Chapter we present some theoretical results for 1-cohomology calculations of the form $H^1(SL_2(k), V_\alpha)_{\omega_r}$ where V_α is the unipotent radical of a minimal parabolic subgroup of reductive G.

Let G be a reductive group over an algebraically closed field k of characteristic $p > 0$, and let B be a Borel subgroup of G containing a maximal torus T of G. This amounts to fixing a base Δ of Φ, where Φ is the root system for G. Then Φ yields a set of representative parabolic subgroups P_I, $I \subset \Delta$ as in [10, §30]. Denote by Φ^+ the positive roots of G.

5.1 Minimal Parabolic Subgroups

Let $P_\alpha < G$ be the parabolic subgroup of G corresponding to the simple root $\alpha \in \Delta$, with Levi subgroup L_α and unipotent radical V_α. We fix an ordering on the roots so that products such as

$$V_\alpha = \prod_{\delta \in \Phi^+ - \{\alpha\}} U_\delta,$$

are unambiguous.

There exists a homomorphism ω_0 from $SL_2(k)$ into L_α such that

$$\omega_0 \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} = \epsilon_\alpha(u),$$
$$\omega_0 \begin{pmatrix} 1 & 0 \\ u & 1 \end{pmatrix} = \epsilon_{-\alpha}(u),$$

where $\epsilon_\alpha : k \to U_\alpha$ is an isomorphism of algebraic groups [10, Theorem 26.3(c)]. Any such isomorphism satisfies

$$t\epsilon_\alpha(x)t^{-1} = \epsilon_\alpha(\alpha(t)x),$$

for all $t \in T, x \in k$.

We fix an integer $r > 0$ and define $\omega_r : SL_2(k) \to L_\alpha$ to be the composition of ω_0 and the Frobenius map, defined by

$$F_r : SL_2(k) \to SL_2(k)$$
$$(A_{ij}) \mapsto (A_{ij})^{p^r}.$$

That is $\omega_r = \omega_0 \circ F_r$ and satisfies

$$\omega_r \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} = \epsilon_\alpha(u^{p^r})$$
$$\omega_r \begin{pmatrix} 1 & 0 \\ u & 1 \end{pmatrix} = \epsilon_{-\alpha}(u^{p^r}).$$

Now we have an action of $SL_2(k)$ on V_α defined by

$$y \cdot v = \omega_r(y)v\omega_r(y^{-1}),$$

for $y \in SL_2(k), v \in V_\alpha$. By [10, Theorem 26.3(c)]

$$\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \cdot \prod_{\delta \in \Phi^+ - \{\alpha\}} \epsilon_\delta(\lambda_\delta) = \prod_{\delta \in \Phi^+ - \{\alpha\}} \epsilon_\delta \left(t^{\langle \delta, \alpha \rangle p^r} \lambda_\delta \right), \tag{5.1}$$

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \cdot \prod_{\delta \in \Phi^+ - \{\alpha\}} \epsilon_\delta(\lambda_\delta) = \prod_{\delta \in \Phi^+ - \{\alpha\}} n_\alpha \epsilon_\delta \left(\lambda_\delta\right) n_\alpha^{-1}, \tag{5.2}$$

for all $t \in k^*, \lambda_\delta \in k$, where $n_\alpha = \epsilon_\alpha(1)\epsilon_{-\alpha}(-1)\epsilon_\alpha(1)$.

We set out to find the general form of a 1-cocycle $\tau \in Z^1(SL_2(k), V_\alpha)_{\omega_r}$. Then we can apply the canonical projection $\psi : Z^1(SL_2(k), V_\alpha)_{\omega_r} \to H^1(SL_2(k), V_\alpha)_{\omega_r}$ to calculate the 1-cohomology. By Lemma 3.28, there exists $\sigma \in Z^1(SL_2(k), V_\alpha)_{\omega_r}$ such that $\psi(\tau) = \psi(\sigma)$ and $\sigma(y) = 1$ for all $y \in T_2(k)$. Below, we assume σ is an element of $Z^1(SL_2(k), V_\alpha)_{\omega_r}$ such that $\sigma(y) = 1$ for all $y \in T_2(k)$.

As a starting point, we have the following Proposition.

Proposition 5.1.

$$\sigma \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} = \prod_{\delta \in \mathcal{D}} \epsilon_\delta \left(x_\delta \left(u \right) \right),$$

where $\mathcal{D} = \{\delta \in \Phi^+ - \{\alpha\} \,|\, \langle \delta, \alpha \rangle > 0\}$, *and the* $x_\delta \in k[X]$ *are polynomials in one variable which satisfy*

$$x_\delta \left(t^2 u \right) = t^{\langle \delta, \alpha \rangle p^r} x_\delta \left(u \right), \tag{5.3}$$

for all $t \in k^*, u \in k$.

Proof. Let $\delta \in \Phi^+ - \{\alpha\}$. We have the chain of morphisms

$$k \simeq U_2(k) \xrightarrow{\iota} SL_2(k) \xrightarrow{\sigma} V_\alpha \xrightarrow{\pi_\delta} k,$$

where ι is the inclusion map and π_δ is the projection onto the root subgroup U_δ. Hence $x_\delta = \pi_\delta \circ \sigma \circ \iota$ is a morphism from $k \to k$.

We apply σ to the identity

$$\begin{pmatrix} 1 & t^2 u \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \begin{pmatrix} t^{-1} & 0 \\ 0 & t \end{pmatrix},$$

using the 1-cocycle condition on the right to obtain

$$\begin{aligned}
\sigma \begin{pmatrix} 1 & t^2 u \\ 0 & 1 \end{pmatrix} &= \sigma \left(\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \begin{pmatrix} t^{-1} & 0 \\ 0 & t \end{pmatrix} \right) \\
&= \left(\sigma \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \right) \left[\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \cdot \sigma \left(\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \begin{pmatrix} t^{-1} & 0 \\ 0 & t \end{pmatrix} \right) \right] \\
&= \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \cdot \left[\left(\sigma \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \right) \left[\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \cdot \sigma \begin{pmatrix} t^{-1} & 0 \\ 0 & t \end{pmatrix} \right] \right] \\
&= \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \cdot \sigma \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}.
\end{aligned}$$

So by Equation 5.1, $x_\delta\left(t^2 u\right) = t^{\langle\delta,\alpha\rangle p^r} x_\delta\left(u\right)$, as claimed in Equation 5.3.

Since x_δ is a polynomial function there can only be non-negative powers of t on the right-hand side of Equation 5.3, which forces $\langle\delta,\alpha\rangle \geq 0$. However, if $\langle\delta,\alpha\rangle = 0$ then x_δ is constant and hence zero, as $\sigma\left(y\right) = 1$ for $y \in T_2(k)$ implies that $x_\delta(0) = 0$. Therefore the non-zero x_δ occur only when $\langle\delta,\alpha\rangle > 0$. □

Remark 5.2. Using a similar argument we see that

$$\sigma\begin{pmatrix} 1 & 0 \\ u & 1 \end{pmatrix} = \prod_{\delta\in\mathcal{D}^-} \epsilon_\delta\left(x_\delta(u)\right),$$

where $\mathcal{D}^- = \{\delta \in \Phi^+ - \{\alpha\} \,|\, \langle\delta,\alpha\rangle < 0\}$, and the $x_\delta \in k[X]$ are polynomials in one variable which satisfy

$$x_\delta(t^{-2}u) = t^{\langle\delta,\alpha\rangle p^r} x_\delta(u).$$

Proposition 5.3. *The images of $U_2(k)$ and $B_2(k)$ under σ are equal.*

Proof. Let $a \in k^*, b \in k$. Since

$$\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}\begin{pmatrix} 1 & a^{-1}b \\ 0 & 1 \end{pmatrix},$$

we get

$$\begin{aligned}
\sigma\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} &= \left(\sigma\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}\right)\left(\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}\cdot\sigma\begin{pmatrix} 1 & a^{-1}b \\ 0 & 1 \end{pmatrix}\right) \\
&= \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}\cdot\sigma\begin{pmatrix} 1 & a^{-1}b \\ 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}\cdot\prod_{\delta\in\mathcal{D}} \epsilon_\delta\left(x_\delta(a^{-1}b)\right) \quad \text{(Proposition 5.1)} \\
&= \prod_{\delta\in\mathcal{D}} \epsilon_\delta\left(a^{\langle\delta,\alpha\rangle p^r} x_\delta(a^{-1}b)\right) \quad \text{(by Equation 5.1)} \\
&\subset \sigma\left(U_2(k)\right)
\end{aligned}$$

The reverse inclusion is obvious. □

Next we prove some useful facts about root systems not containing G_2 or C_3.

Proposition 5.4. *Suppose Φ does not contain G_2 and let $\alpha, \beta \in \Phi$. If $\alpha + \beta \in \Phi$ then $\langle\alpha,\beta\rangle \leq 0$.*

Proof. We have

$$\langle\alpha,\beta\rangle > 0 \Leftrightarrow (\alpha,\beta) > 0 \Leftrightarrow \cos(\theta) > 0,$$

where θ is the angle between α and β. Hence acute angles correspond to positive pairs. Since α,β lie in a rank 2 root subsystem of Φ [10, A.4], we only need to refer to the root system diagrams for $A_1 \times A_1, A_2, B_2$ (Appendix B.3) to see that no two roots meeting at an acute angle add to give another root. Therefore, if $\langle\alpha,\beta\rangle > 0$ then $\alpha+\beta \notin \Phi$. $\square$

Remark 5.5. If $\Phi = G_2$, $\Delta = \{\alpha,\beta\}$ (α short) then $\alpha, 2\alpha+\beta, 3\alpha+\beta \in \Phi$ and $\langle\alpha, 2\alpha+\beta\rangle = 1$. This shows that Proposition 5.4 breaks down when $\Phi = G_2$.

Remark 5.6. The only irreducible root system containing G_2 is G_2 itself.

Proposition 5.7. *Suppose Φ does not contain G_2 or C_3. Let $\delta_1,\delta_2 \in \Phi$ and $\gamma \in \Delta$ be distinct roots such that $\langle\delta_i,\gamma\rangle > 0$ $(i = 1,2)$. If $\delta_1+\delta_2$ is a root, then δ_1 and δ_2 are of opposite sign.*

Proof. Suppose $\delta_1 + \delta_2 \in \Phi$. Let θ_i be the absolute value of the angle between δ_i and γ $(i = 1,2)$, and let θ_3 be the absolute value of the angle between δ_1 and δ_2. Then

$$\begin{aligned}
&\langle\delta_i,\gamma\rangle > 0\\
\Rightarrow &(\delta_i,\gamma) > 0\\
\Rightarrow &\cos(\theta_i) > 0\\
\Rightarrow &\theta_i < \pi/2, \quad (i = 1,2).
\end{aligned}$$

By Proposition 5.4 $\langle\delta_1,\delta_2\rangle \le 0$, so $\theta_3 \ge \pi/2$.

Without loss of generality, this leads to the following four cases to consider.

(i) $\theta_1 = \pi/3, \quad \theta_2 = \pi/3, \quad \theta_3 = 2\pi/3,$

(ii) $\theta_1 = \pi/4, \quad \theta_2 = \pi/4, \quad \theta_3 = \pi/2,$

(iii) $\theta_1 = \pi/4, \quad \theta_2 = \pi/3, \quad \theta_3 = \pi/2,$

(iv) $\theta_1 = \pi/3, \quad \theta_2 = \pi/3, \quad \theta_3 = \pi/2,$

For cases (i) and (ii) the three roots must lie in a plane, hence in a rank 2 subsystem of Φ. Since we ruled out $\Phi = G_2$, this leaves $A_1 \times A_1, A_2$, or B_2. In fact, they lie in A_2 and B_2 subsystems, respectively. Consulting the root system diagrams for each case (Appendix B.3) we see that if $\gamma = \delta_1+\delta_2$ then δ_1 and δ_2 are of opposite sign, as claimed.

Like case (ii), for case (iii) δ_1, δ_2 must lie in a B_2 subsystem, and they must have the same length. (Note that $A_1 \times A_1$ is ruled out because $\delta_1 + \delta_2$ is a root). However, $\theta_1 = \pi/4$ implies that δ_1, γ are roots of different length in a B_2 subsystem, while $\theta_2 = \pi/3$ implies that δ_2, γ are roots of the same length in an A_2 subsystem. Hence δ_1 and δ_2 have different lengths, a contradiction. Therefore we rule out case (iii).

For case (iv) δ_1, δ_2 again lie in a B_2 subsystem, so δ_1 and δ_2 are short roots because $\delta_1 + \delta_2$ is a root. Furthermore δ_i, γ lie in an A_2 subsystem ($i = 1, 2$), so the roots $\delta_1, \delta_2, \gamma$ are the same length. Hence γ is a short root. Therefore, if $\delta_1, \delta_2, \gamma$ lie in a rank 3 subsystem, this subsystem contains two short root A_2 subsystems and a B_2 subsystem. This rules out B_3, since B_3 has only three positive short roots, none of which lies in an A_2 subsystem (pairwise, they lie in one of three B_2 subsystems). Since we excluded C_3, we can rule out case (iv). □

Remark 5.8. Let $\Phi = C_3$, $\Delta = \{\alpha, \beta, \gamma\}$ where γ is the long root and β is the short root connected to γ. Then

$$\begin{aligned}\langle \alpha + \beta, \alpha \rangle &= 1,\\ \langle \alpha + \beta + \gamma, \alpha \rangle &= 1,\\ (\alpha + \beta) + (\alpha + \beta + \gamma) &= 2\alpha + 2\beta + \gamma \in \Phi, \text{ and}\\ \alpha + \beta, \alpha + \beta + \gamma &\in \Phi^+.\end{aligned}$$

This shows that Proposition 5.7 breaks down when Φ contains C_3 as a root subsystem.

Remark 5.9. Unlike the case for G_2 (Remark 5.6), it is not obvious whether or not an irreducible root system contains C_3.

Lemma 5.10 (First Main Lemma)**.** *Suppose Φ does not contain G_2 or C_3. Then*

(i) $y_1 \cdot \sigma(y_2) = \sigma(y_2)$, for all $y_1, y_2 \in U_2(k)$, and

(ii) $\sigma(B_2(k))$ lies in a product of commuting root groups of V_α.

Proof. Let $u_1, u_2 \in k$. By Proposition 5.1

$$\begin{pmatrix} 1 & u_1 \\ 0 & 1 \end{pmatrix} \cdot \sigma \begin{pmatrix} 1 & u_2 \\ 0 & 1 \end{pmatrix} = \epsilon_\alpha(u_1^{p^r}) \left[\prod_{\delta \in \mathcal{D}} \epsilon_\delta \left(x_\delta \left(u_2 \right) \right) \right] \epsilon_\alpha(-u_1^{p^r}),$$

where $\mathcal{D} = \{\delta \in \Phi^+ - \{\alpha\} \mid \langle \delta, \alpha \rangle > 0\}$.

Let $\delta \in \mathcal{D}$. Since $\langle \delta, \alpha \rangle > 0$, $\alpha + \delta$ is not a root (Proposition 5.4), so U_α commutes with U_δ. This proves (i).

Let $\delta_1, \delta_2 \in \mathcal{D}$. Since $\langle \delta_i, \alpha \rangle > 0$ $(i = 1, 2)$, $\delta_1 + \delta_2$ is not a root (Proposition 5.7), so U_{δ_1} and U_{δ_2} commute. Therefore $\sigma\left(U_2(k)\right)$ lies in a product of commuting root groups, so $\sigma\left(B_2(k)\right)$ lies in a product of commuting root groups (Proposition 5.3). □

Lemma 5.11 (Second Main Lemma)**.** *Let $\sigma \in Z^1(SL_2(k), V_\alpha)_{\omega_r}$ such that $\sigma(y) = 1$ for all $y \in T_2(k)$ and suppose*

(i) $y_1 \cdot \sigma(y_2) = \sigma(y_2)$, for all $y_1, y_2 \in U_2(k)$, and

(ii) $\sigma\left(U_2(k)\right)$ lies in a product of commuting root groups of V_α.

Then for all $\delta \in \mathcal{D}$, there exist $\mu_\delta \in k$ such that for all $a \in k^*$ and all $b \in k$

$$\sigma \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} = \prod_{\delta \in \mathcal{D}} \epsilon_\delta \left(a^{(\langle \delta, \alpha \rangle p^r - n(\delta))} b^{n(\delta)} \mu_\delta \right),$$

where $n(\delta) = p^{r-2+\langle \delta, \alpha \rangle}$, and $\mathcal{D}$ is defined by

$$\mathcal{D} = \begin{cases} \{\delta \in \Phi^+ - \{\alpha\} \mid \langle \delta, \alpha \rangle = 1 \text{ or } 2\}, & \text{if } p = 2 \\ \\ \{\delta \in \Phi^+ - \{\alpha\} \mid \langle \delta, \alpha \rangle = 2\}, & \text{otherwise.} \end{cases}$$

Furthermore, if $\langle \delta, \alpha \rangle = 1$ and $\sigma(K) \cap U_\delta$ is non-trivial for some $\delta \in \mathcal{D}$ then $r > 0$.

Proof. We apply σ to both sides of the equation

$$\begin{pmatrix} 1 & u_1 + u_2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & u_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & u_2 \\ 0 & 1 \end{pmatrix},$$

to get

$$\begin{aligned} \sigma \begin{pmatrix} 1 & u_1 + u_2 \\ 0 & 1 \end{pmatrix} &= \left(\sigma \begin{pmatrix} 1 & u_1 \\ 0 & 1 \end{pmatrix} \right) \left(\begin{pmatrix} 1 & u_1 \\ 0 & 1 \end{pmatrix} \cdot \sigma \begin{pmatrix} 1 & u_2 \\ 0 & 1 \end{pmatrix} \right) \\ &= \sigma \begin{pmatrix} 1 & u_1 \\ 0 & 1 \end{pmatrix} \sigma \begin{pmatrix} 1 & u_2 \\ 0 & 1 \end{pmatrix} \quad \text{(by (i))}. \end{aligned} \tag{5.4}$$

By Proposition 5.1 there exist polynomials x_δ such that

$$\sigma \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} = \prod_{\delta \in \mathcal{D}} \epsilon_\delta \left(x_\delta(u) \right).$$

Applying Equation 5.4 yields

$$\begin{aligned}\prod_{\delta\in\mathcal{D}} \epsilon_\delta(x_\delta(u_1+u_2)) &= \prod_{\delta\in\mathcal{D}} \epsilon_\delta(x_\delta(u_1)) \prod_{\delta\in\mathcal{D}} \epsilon_\delta(x_\delta(u_2)) \\ &= \prod_{\delta\in\mathcal{D}} \epsilon_\delta(x_\delta(u_1))\epsilon_\delta(x_\delta(u_2)) \quad \text{(by (ii))} \\ &= \prod_{\delta\in\mathcal{D}} \epsilon_\delta(x_\delta(u_1)+x_\delta(u_2)).\end{aligned}$$

Hence each x_δ is an additive polynomial, so it is of the form

$$x_\delta(\lambda) = \sum_{i=0}^{n} \mu_i \lambda^{p^i} \quad \text{([10, §20.3, Lemma A])},$$

for some $n \in \mathbb{N}$ and some $\mu_i \in k$ depending on δ. Then by Equation 5.3 of Proposition 5.1

$$\sum_{i=0}^{n} \mu_i (t^2 u)^{p^i} = t^{\langle\delta,\alpha\rangle p^r} \sum_{i=0}^{n} \mu_i u^{p^i},$$

for all $t \in k^*, u \in k$.

Suppose $x_\delta \neq 0$, so there exists $j \geq 0$ such that $\mu_j \neq 0$. Then

$$\begin{aligned}\mu_j t^{2p^j} u^{p^j} &= t^{\langle\delta,\alpha\rangle p^r} \mu_j u^{p^j} \\ \Rightarrow t^{2p^j} &= t^{\langle\delta,\alpha\rangle p^r} \\ \Rightarrow 2p^j &= \langle\delta,\alpha\rangle p^r. \end{aligned} \tag{5.5}$$

In [17, §3.4] it is shown that the possible pairings of any two roots are bounded by ± 3. Hence by Proposition 5.1 $\langle\delta,\alpha\rangle = 1, 2$ or 3. Therefore the only solutions to Equation 5.5 are

$$\begin{aligned} &p = 2, \quad \langle\delta,\alpha\rangle = 1, \quad j = r-1, \\ &p \geq 2, \quad \langle\delta,\alpha\rangle = 2, \quad j = r, \end{aligned}$$

hence the lower bound on r in the statement of the Lemma. This shows that it is only possible for x_δ to have one nonzero term, either the $(r-1)^{th}$ or the r^{th} depending on whether $\langle\delta,\alpha\rangle = 1$ or 2, respectively. Furthermore, the $(r-1)^{th}$ term is zero if $p \neq 2$.

Let $\mu_\delta = \mu_j$ and let $n(\delta) = p^{r-2+\langle\delta,\alpha\rangle}$, so that

$$\sigma \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} = \prod_{\delta\in\mathcal{D}'} \epsilon_\delta\left(u^{n(\delta)}\mu_\delta\right),$$

where

$$\mathcal{D}' = \begin{cases} \{\delta \in \Phi^+ \mid \langle \delta, \alpha \rangle = 1 \text{ or } 2\}, & \text{if } p = 2, \\ \\ \{\delta \in \Phi^+ \mid \langle \delta, \alpha \rangle = 2\}, & \text{otherwise.} \end{cases}$$

Furthermore

$$\begin{aligned} \sigma \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} &= \sigma \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \left[\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \cdot \sigma \begin{pmatrix} 1 & a^{-1}b \\ 0 & 1 \end{pmatrix} \right] \\ &= \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \cdot \sigma \begin{pmatrix} 1 & a^{-1}b \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \cdot \prod_{\delta \in \mathcal{D}'} \epsilon_\delta \left((a^{-1}b)^{n(\delta)} \mu_\delta \right) \\ &= \prod_{\delta \in \mathcal{D}'} \epsilon_\delta \left(a^{\langle \delta, \alpha \rangle p^r} (a^{-1}b)^{n(\delta)} \mu_\delta \right) \quad \text{(Equation 5.1)} \\ &= \prod_{\delta \in \mathcal{D}'} \epsilon_\delta \left(a^{(\langle \delta, \alpha \rangle p^r - n(\delta))} b^{n(\delta)} \mu_\delta \right). \end{aligned}$$

This completes the proof. □

Remark 5.12. Suppose that

(i′) $y_1 \cdot \sigma(y_2) = \sigma(y_2)$, for all $y_1, y_2 \in U_2^-(k)$, and

(ii′) $\sigma(U_2^-(k))$ lies in a product of commuting root groups of V_α.

Then for all $\delta \in \mathcal{D}^-$, there exist $\nu_\delta \in k$ such that for all $a \in k^*$ and all $c \in k$

$$\sigma \begin{pmatrix} a & 0 \\ c & a^{-1} \end{pmatrix} = \prod_{\delta \in \mathcal{D}^-} \epsilon_\delta \left(a^{(\langle \delta, \alpha \rangle p^r + n(\delta))} c^{n(\delta)} \nu_\delta \right),$$

where $n(\delta) = p^{r-2-\langle \delta, \alpha \rangle}$, and $\mathcal{D}^-$ is defined by

$$\mathcal{D}^- = \begin{cases} \{\delta \in \Phi^+ - \{\alpha\} \mid \langle \delta, \alpha \rangle = -1 \text{ or } -2\}, & \text{if } p = 2, \\ \\ \{\delta \in \Phi^+ - \{\alpha\} \mid \langle \delta, \alpha \rangle = -2\}, & \text{otherwise.} \end{cases}$$

The proof is similar to the proof of Lemma 5.11, replacing α with $-\alpha$.

Corollary 5.13. *Suppose σ satisfies the hypotheses of Lemma 5.11. Then*

$$\sigma \begin{pmatrix} -a & -b \\ 0 & -a^{-1} \end{pmatrix} = \sigma \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}.$$

Proof. It is obvious for $p = 2$. Suppose $p > 2$, then

$$\begin{aligned}\sigma\begin{pmatrix}-a & -b\\ 0 & -a^{-1}\end{pmatrix} &= \prod_{\delta\in\mathcal{D}} \epsilon_\delta\left((-a)^{2p^r-p^r}(-b)^{p^r}\mu_\delta\right)\\ &= \prod_{\delta\in\mathcal{D}} \epsilon_\delta\left((-a)^{p^r}(-b)^{p^r}\mu_\delta\right)\\ &= \prod_{\delta\in\mathcal{D}} \epsilon_\delta\left((ab)^{p^r}\mu_\delta\right)\\ &= \sigma\begin{pmatrix}a & b\\ 0 & a^{-1}\end{pmatrix}.\end{aligned}$$

□

5.2 Discussion

In the next Chapter we investigate the counterexamples to Propositions 5.4 and 5.7 in the respective Remarks 5.5 and 5.8 by deriving the form of $\sigma \in Z^1(SL_2(k), V_\alpha)_{\omega_r}$ for the appropriate simple roots $\alpha \in \Delta$. This is carried out in Sections 6.3.1 and 6.4.1, respectively. We show that σ still satisfies the conclusion of Lemma 5.11. This gives some evidence for the following Conjecture.

Conjecture 5.14. *Let G be a reductive group, P_α a minimal parabolic subgroup of G with unipotent radical V_α. Let $x \in H^1(SL_2(k), V_\alpha)_{\omega_r}$. Then there exists $\sigma \in Z^1(SL_2(k), V_\alpha)_{\omega_r}$ such that $\psi(\sigma) = x$. Furthermore, there exist $\mu_\delta \in k$ such that for all $a \in k^*$, $b \in k$*

$$\sigma\begin{pmatrix}a & b\\ 0 & a^{-1}\end{pmatrix} = \prod_{\delta\in\mathcal{D}} \epsilon_\delta\left(a^{(\langle\delta,\alpha\rangle p^r - n(\delta))}b^{n(\delta)}\mu_\delta\right),$$

where $n(\delta) = p^{r-2+\langle\delta,\alpha\rangle}$, and $\mathcal{D}$ is defined by

$$\mathcal{D} = \begin{cases}\{\delta \in \Phi^+ \mid \langle\delta,\alpha\rangle = 1 \text{ or } 2\}, & \text{if } p = 2,\\ \{\delta \in \Phi^+ \mid \langle\delta,\alpha\rangle = 2\}, & \text{otherwise.}\end{cases}$$

If the above Conjecture is true, it would be worthwhile to see whether it can be extended to all parabolic subgroups P_I, $I \subset \Delta$ of G. This would be useful for further 1-cohomology calculations following the method we employ in the next Chapter.

Furthermore, given that $H^1(SL_2(k), V_\alpha)_{\omega_r} \to H^1(B_2(k), V_\alpha)_{\omega_r}$ is already injective by Lemma 3.27 and $\sigma\,(B_2(k))$ lies in an abelian subgroup of V_α, it may be possible to show that the restriction map

$$H^1(SL_2(k), V_\alpha)_{\omega_r} \to H^1(U_2(k), V_\alpha)_{\omega_r},$$

is injective (cf. Example 3.2). Thus there is evidence that the answer to the algebraic version of Külshammer's second question for $SL_2(k)$ and connected reductive G is "yes" (cf. Theorem 4.22).

Chapter 6

Example 1-Cohomology Calculations

In this Chapter we calculate some examples of 1-cohomology.

We calculate the 1-cohomology of B_2, aided by the results in the previous Chapter. We also investigate the counterexamples to Propositions 5.4 and 5.7 as pointed out in the respective Remarks 5.5 and 5.8 of the previous Chapter.

In our last example calculation we use the 1-cohomology to find a family of maps from $SL_2(k)$ to B_4 which gives infinitely many conjugacy classes. We would like to say whether or not this leads to a counterexample to Külshammer's second question however the calculation is not developed enough for us to apply Theorem 4.22.

6.1 General Method

The general method we use of calculating the 1-cohomology is outlined below.

For each $\alpha \in \Delta$ (the simple roots)

1. Fix $x \in H^1(SL_2(k), V_\alpha)_{\omega_r}$. Then there exists $\sigma \in Z^1(SL_2(k), V_\alpha)_{\omega_r}$ such that $\psi(\sigma) = x$ and $\sigma\,(y) = 1 \in V_\alpha$ for all $y \in T_2(k)$ (Lemma 3.28).
2. Use Lemma 5.11 to determine $\sigma \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}$.
3. If $\sigma\,(y) = 1$ for all $y \in B_2(k)$ then $H^1(SL_2(k), V_\alpha)_{\omega_r}$ is trivial by Lemma 3.27.

4. Otherwise, get a formula for $\sigma\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ for $c \neq 0$ by

$$\sigma\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \sigma\left(\begin{pmatrix} 1 & ac^{-1} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} -c & -d \\ 0 & -c^{-1} \end{pmatrix}\right).$$

5. Show that the formula for σ in step 4 gives a well-defined 1-cocycle on $SL_2(k)$.

6. Calculate $H^1(SL_2(k), V_\alpha)_{\omega_r}$.

7. Calculate $H^1(SL_2(k), V_\alpha)_{\omega_r}/C_L(\omega_r)$ and use Theorem 4.22 to get information about conjugacy classes in $\mathrm{Hom}(SL_2(k), P_\alpha)$.

We elaborate on steps 4 and 5. Given $\sigma\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}$ and $\sigma\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, the obvious way to piece together a 1-cocycle on $SL_2(k)$ is, for $c \neq 0$

$$\begin{aligned}
\sigma\begin{pmatrix} a & b \\ c & d \end{pmatrix} &= \sigma\left(\begin{pmatrix} 1 & ac^{-1} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} -c & -d \\ 0 & -c^{-1} \end{pmatrix}\right) \\
&= \sigma\begin{pmatrix} 1 & ac^{-1} \\ 0 & 1 \end{pmatrix}\left[\begin{pmatrix} 1 & ac^{-1} \\ 0 & 1 \end{pmatrix} \cdot \sigma\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} -c & -d \\ 0 & -c^{-1} \end{pmatrix}\right)\right] \\
&= \sigma\begin{pmatrix} 1 & ac^{-1} \\ 0 & 1 \end{pmatrix}\left[\begin{pmatrix} 1 & ac^{-1} \\ 0 & 1 \end{pmatrix} \cdot \left(\sigma\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\left[\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \cdot \sigma\begin{pmatrix} -c & -d \\ 0 & -c^{-1} \end{pmatrix}\right]\right)\right] \\
&= \sigma\begin{pmatrix} 1 & ac^{-1} \\ 0 & 1 \end{pmatrix}\left[\begin{pmatrix} 1 & ac^{-1} \\ 0 & 1 \end{pmatrix} \cdot \left(\sigma\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\left[\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \cdot \sigma\begin{pmatrix} c & d \\ 0 & c^{-1} \end{pmatrix}\right]\right)\right],
\end{aligned} \tag{6.1}$$

where we apply Corollary 5.13 at the last step.

This gives a candidate for a 1-cocycle on $SL_2(k)$. In [18, Proposition 2] Martin establishes a criterion for a function to belong to the set of 1-cocycles in terms of group presentations. We can use this criterion to prove that our candidate σ is a well-defined 1-cocycle on $SL_2(k)$ based on Steinberg's presentation for SL_2 [17, §12.1]. This amounts to verifying the following equations hold.

$$\sigma(x_r(t_1))\,[x_r(t_1) \cdot \sigma(x_r(t_2))] = \sigma(x_r(t_1 + t_2)), \tag{6.2}$$

$$\sigma(h_r(t_1))\,[h_r(t_1) \cdot \sigma(h_r(t_2)))] = \sigma(h_r(t_1 t_2)), \tag{6.3}$$

$$\sigma(n_r(t))\,\big[n_r(t) \cdot \big[\sigma(x_r(u))\,\big[x_r(u) \cdot \sigma(n_r(t)^{-1})\big]\big]\big] = \sigma(x_{-r}(-t^{-2}u)), \tag{6.4}$$

for $r \in \{\pm\alpha\}$, where

$$
\begin{aligned}
x_\alpha(t) &= \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}, \\
x_{-\alpha}(t) &= \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}, \\
n_r(t) &= x_r(t)x_{-r}(-t^{-1})x_r(t), \text{ and} \\
h_r(t) &= n_r(t)n_r(-1).
\end{aligned}
$$

The checks are routine and lengthy, so we omit the details in our examples that follow. Note that not all of the equations need to be checked, for example Equation 6.2 is automatic for $r = \alpha$, since σ is already a well-defined 1-cocycle on $B_2(k)$ at this stage.

We will need to determine the action of $SL_2(k)$ on V_α and the group law on V_α which often involves lengthy computations using the commutator relations. For clarity, we will use column vector notation.

$$
\begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} = \epsilon_{\delta_1}(v_1)\epsilon_{\delta_2}(v_2)\cdots\epsilon_{\delta_n}(v_n) \in V_\alpha
$$

Determining the values of σ in Equation 6.1 are straightforward. Lemma 5.11 gives $\sigma\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}$ up to some constants $\mu_\delta \in k$, while $\sigma\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ is computed as follows.

Proposition 6.1. *Let σ satisfy the hypotheses of Lemma 5.11. Then there exist constants $\lambda_\delta \in k$ such that*

$$
\begin{aligned}
\sigma\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} &= \sigma\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}\left[\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}\cdot\left(\sigma\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\left[\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\cdot\sigma\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}\right]\right)\right] && (6.5) \\
&= \prod_{\delta\in\mathcal{D}} \epsilon_\delta(\lambda_\delta), && (6.6)
\end{aligned}
$$

where $\mathcal{D} = \{\delta \in \Phi^+ - \{\alpha\} \mid \langle\delta, \alpha\rangle = 0\}$.

Proof. Since

$$
\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} t^{-1} & 0 \\ 0 & t \end{pmatrix},
$$

we have

$$\begin{aligned}
&\sigma\left(\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right) = \sigma\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} t^{-1} & 0 \\ 0 & t \end{pmatrix}\right) \\
\Rightarrow &\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \cdot \sigma\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \sigma\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.
\end{aligned}$$

Therefore $\sigma\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ is fixed under the action of $T_2(k)$ and so is of the form

$$\sigma\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \prod_{\delta \in \mathcal{D}} \epsilon_\delta(\lambda_\delta),$$

for some $\lambda_\delta \in k$, where

$$\mathcal{D} = \{\delta \in \Phi^+ - \{\alpha\} \mid \langle \delta, \alpha \rangle = 0\}.$$

Furthermore, since

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix},$$

we have

$$\begin{aligned}
\sigma\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} &= \sigma\left(\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}\right) \\
&= \sigma\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}\left[\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \cdot \sigma\left(\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}\right)\right] \\
&= \sigma\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}\left[\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \cdot \left(\sigma\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\left[\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \cdot \sigma\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}\right]\right)\right].
\end{aligned}$$

□

6.2 $G = B_2$

Let T be a maximal torus of B_2 over an algebraically closed field k of characteristic p. We label the positive roots for B_2 as $\alpha, \beta, \alpha+\beta, 2\alpha+\beta$. We have from [10, §33.4]:

$$\begin{aligned}
\epsilon_\beta(y)\epsilon_\alpha(x) &= \epsilon_\alpha(x)\epsilon_\beta(y)\epsilon_{\alpha+\beta}(xy)\epsilon_{2\alpha+\beta}(x^2y) \\
\epsilon_{\alpha+\beta}(y)\epsilon_\alpha(x) &= \epsilon_\alpha(x)\epsilon_{\alpha+\beta}(y)\epsilon_{2\alpha+\beta}(2xy),
\end{aligned}$$

and

$$\begin{aligned}
n_\alpha \epsilon_\beta(x) n_\alpha^{-1} &= \epsilon_{2\alpha+\beta}(x) \\
n_\alpha \epsilon_{\alpha+\beta}(x) n_\alpha^{-1} &= \epsilon_{\alpha+\beta}(-x) \\
n_\alpha \epsilon_{2\alpha+\beta}(x) n_\alpha^{-1} &= \epsilon_\beta(x) \\
n_\beta \epsilon_\alpha(x) n_\beta^{-1} &= \epsilon_{\alpha+\beta}(x) \\
n_\beta \epsilon_{\alpha+\beta}(x) n_\beta^{-1} &= \epsilon_\alpha(-x) \\
n_\beta \epsilon_{2\alpha+\beta}(x) n_\beta^{-1} &= \epsilon_{2\alpha+\beta}(x)
\end{aligned}$$

A proper parabolic subgroup of B_2 is conjugate to one of

$$\begin{aligned}
P_\alpha &= \langle B, U_{-\alpha} \rangle \\
P_\beta &= \langle B, U_{-\beta} \rangle,
\end{aligned}$$

where B is the Borel subgroup of B_2 containing T

$$B = \langle T, U_\alpha, U_\beta, U_{\alpha+\beta}, U_{2\alpha+\beta} \rangle.$$

The two parabolic subgroups have the Levi decompositions

$$\begin{aligned}
P_\alpha &= R_u(P_\alpha) \rtimes L_\alpha \\
&= \langle U_\beta, U_{\alpha+\beta}, U_{2\alpha+\beta} \rangle \rtimes \langle T, U_\alpha, U_{-\alpha} \rangle \\
P_\beta &= R_u(P_\beta) \rtimes L_\beta \\
&= \langle U_\alpha, U_{\alpha+\beta}, U_{2\alpha+\beta} \rangle \rtimes \langle T, U_\beta, U_{-\beta} \rangle
\end{aligned}$$

6.2.1 $V = R_u(P_\alpha)$

Let $V_\alpha = R_u(P_\alpha) = \langle U_\beta, U_{\alpha+\beta}, U_{2\alpha+\beta} \rangle$. Note that V_α is abelian. We will write $\mathbf{v} \in V_\alpha$ as a column vector for convenience,

$$\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \epsilon_\beta(v_1)\epsilon_{\alpha+\beta}(v_2)\epsilon_{2\alpha+\beta}(v_3),$$

and we compute the following.

$$\begin{aligned}
\begin{pmatrix}1 & u\\ 0 & 1\end{pmatrix}\cdot \mathbf{v} &= \omega_r\begin{pmatrix}1 & u\\ 0 & 1\end{pmatrix}\mathbf{v}\left(\omega_r\begin{pmatrix}1 & u\\ 0 & 1\end{pmatrix}\right)^{-1}\\
&= \epsilon_\alpha(u^{p^r})\epsilon_\beta(v_1)\epsilon_{\alpha+\beta}(v_2)\epsilon_{2\alpha+\beta}(v_3)\epsilon_\alpha(-u^{p^r})\\
&= \epsilon_\alpha(u^{p^r})\epsilon_\beta(v_1)\epsilon_{\alpha+\beta}(v_2)\epsilon_\alpha(-u^{p^r})\epsilon_{2\alpha+\beta}(v_3)\\
&= \epsilon_\alpha(u^{p^r})\epsilon_\beta(v_1)\epsilon_\alpha(-u^{p^r})\epsilon_{\alpha+\beta}(v_2)\epsilon_{2\alpha+\beta}(-2u^{p^r}v_2)\epsilon_{2\alpha+\beta}(v_3)\\
&= \epsilon_\alpha(u^{p^r})\epsilon_\alpha(-u^{p^r})\epsilon_\beta(v_1)\epsilon_{\alpha+\beta}(-u^{p^r}v_1)\epsilon_{2\alpha+\beta}(u^{2p^r}v_1)\epsilon_{\alpha+\beta}(v_2)\epsilon_{2\alpha+\beta}(v_3-2u^{p^r}v_2)\\
&= \epsilon_\beta(v_1)\epsilon_{\alpha+\beta}(v_2-u^{p^r}v_1)\epsilon_{2\alpha+\beta}(v_3-2u^{p^r}v_2+u^{2p^r}v_1)\\
&= \begin{pmatrix} v_1\\ v_2-u^{p^r}v_1\\ v_3-2u^{p^r}v_2+u^{2p^r}v_1\end{pmatrix},
\end{aligned}$$

$$\begin{aligned}
\begin{pmatrix}t & 0\\ 0 & t^{-1}\end{pmatrix}\cdot \mathbf{v} &= \omega_r\begin{pmatrix}t & 0\\ 0 & t^{-1}\end{pmatrix}\mathbf{v}\left(\omega_r\begin{pmatrix}t & 0\\ 0 & t^{-1}\end{pmatrix}\right)^{-1}\\
&= \alpha^\vee(t^{p^r})\epsilon_\beta(v_1)\epsilon_{\alpha+\beta}(v_2)\epsilon_{2\alpha+\beta}(v_3)\left(\alpha^\vee(t^{p^r})\right)^{-1}\\
&= \epsilon_\beta\left(\beta(\alpha^\vee(t^{p^r}))v_1\right)\epsilon_{\alpha+\beta}\left((\alpha+\beta)(\alpha^\vee(t^{p^r}))v_2\right)\epsilon_{2\alpha+\beta}\left((2\alpha+\beta)(\alpha^\vee(t^{p^r}))v_3\right)\\
&= \epsilon_\beta\left(t^{\langle\beta,\alpha\rangle p^r}v_1\right)\epsilon_{\alpha+\beta}\left(t^{\langle\alpha+\beta,\alpha\rangle p^r}v_2\right)\epsilon_{2\alpha+\beta}\left(t^{\langle 2\alpha+\beta,\alpha\rangle p^r}v_3\right)\\
&= \begin{pmatrix} t^{-2p^r}v_1\\ v_2\\ t^{2p^r}v_3\end{pmatrix}
\end{aligned}$$

$$\begin{aligned}
\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\cdot \mathbf{v} &= \omega_r\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\mathbf{v}\left(\omega_r\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\right)^{-1}\\
&= n_\alpha\epsilon_\beta(v_1)\epsilon_{\alpha+\beta}(v_2)\epsilon_{2\alpha+\beta}(v_3)n_\alpha^{-1}\\
&= n_\alpha\epsilon_\beta(v_1)n_\alpha^{-1}n_\alpha\epsilon_{\alpha+\beta}(v_2)n_\alpha^{-1}n_\alpha\epsilon_{2\alpha+\beta}(v_3)n_\alpha^{-1}\\
&= \epsilon_{2\alpha+\beta}(v_1)\epsilon_{\alpha+\beta}(-v_2)\epsilon_\beta(v_3)\\
&= \epsilon_\beta(v_3)\epsilon_{\alpha+\beta}(-v_2)\epsilon_{2\alpha+\beta}(v_1)\\
&= \begin{pmatrix} v_3\\ -v_2\\ v_1\end{pmatrix}.
\end{aligned}$$

This is enough to determine the action of $SL_2(k)$ on V_α. We leave the details to the reader.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \mathbf{v} = \begin{pmatrix} d^{2p^r} v_1 - 2(cd)^{p^r} v_2 + c^{2p^r} v_3 \\ (ad+bc)^{p^r} v_2 - (bd)^{p^r} v_1 - (ac)^{p^r} v_3 \\ b^{2p^r} v_1 - 2(ab)^{p^r} v_2 + a^{2p^r} v_3 \end{pmatrix}$$

Let $x \in H^1(SL_2(k), V_\alpha)_{\omega_r}$. By Proposition 3.28 there exists $\sigma \in Z^1(SL_2(k), V_\alpha)_{\omega_r}$ such that $\sigma(y) = 1$ for all $y \in T_2(k)$ and $\psi(\sigma) = x$. By Lemma 5.10, we can apply Lemma 5.11, which gives

$$\sigma \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} = \epsilon_{2\alpha+\beta} \left(\mu_3 (ab)^{p^r} \right),$$

for all $a \in k^*, b \in k$, for some fixed $\mu_3 \in k$.

Suppose $p \neq 2$, and let $\mathbf{w} = \begin{pmatrix} 0 \\ 2^{-1}\mu_3 \\ 0 \end{pmatrix} \in V_\alpha$. Now consider $\chi_{\mathbf{w}}^{SL_2(k)} \in B^1(SL_2(k), V_\alpha)_{\omega_r}$.

$$\begin{aligned}
\chi_{\mathbf{w}}^{SL_2(k)} \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} &= \mathbf{w} - \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \cdot \mathbf{w} \\
&= \begin{pmatrix} 0 \\ 2^{-1}\mu_3 \\ 0 \end{pmatrix} - \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 2^{-1}\mu_3 \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 \\ 2^{-1}\mu_3 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 2^{-1}\mu_3 \\ -2(ab)^{p^r}(2^{-1}\mu_3) \end{pmatrix} \\
&= \begin{pmatrix} 0 \\ 2^{-1}\mu_3 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 2^{-1}\mu_3 \\ -(ab)^{p^r}\mu_3 \end{pmatrix} \\
&= \begin{pmatrix} 0 \\ 0 \\ (ab)^{p^r}\mu_3 \end{pmatrix} \\
&= \sigma \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}.
\end{aligned}$$

This shows that if $p \neq 2$ then $H^1(B_2(k), V_\alpha)_{\omega_r}$ is trivial, and hence $H^1(SL_2(k), V_\alpha)_{\omega_r}$ is trivial by Lemma 3.27. Therefore, by Theorem 4.19 there are only finitely many P-conjugacy classes of representations from K to P for each choice of ω.

We proceed with $p = 2$. By Lemma 5.11 and Remark 5.12 there exist constants $\mu_1, \mu_3 \in k$ such that

$$\sigma\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \mu_3(ab)^{2^r} \end{pmatrix},$$

$$\sigma\begin{pmatrix} d^{-1} & 0 \\ c & d \end{pmatrix} = \begin{pmatrix} \mu_1(cd)^{2^r} \\ 0 \\ 0 \end{pmatrix}.$$

Furthermore, by Equation 6.6 there exists $\lambda_2 \in k$ such that

$$\sigma\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 \\ \lambda_2 \\ 0 \end{pmatrix},$$

and by Equation 6.5

$$\begin{aligned}
\sigma\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} &= \sigma\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \cdot \left(\sigma\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \cdot \sigma\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\right) \\
&= \begin{pmatrix} 0 \\ 0 \\ \mu_3 \end{pmatrix} + \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \cdot \left(\begin{pmatrix} \mu_1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ \mu_3 \end{pmatrix}\right) \\
&= \begin{pmatrix} 0 \\ 0 \\ \mu_3 \end{pmatrix} + \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \cdot \left(\begin{pmatrix} \mu_1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} \mu_3 \\ \mu_3 \\ \mu_3 \end{pmatrix}\right) \\
&= \begin{pmatrix} 0 \\ 0 \\ \mu_3 \end{pmatrix} + \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \mu_1 + \mu_3 \\ \mu_3 \\ \mu_3 \end{pmatrix} \\
&= \begin{pmatrix} 0 \\ 0 \\ \mu_3 \end{pmatrix} + \begin{pmatrix} \mu_1 + \mu_3 \\ \mu_3 + \mu_1 + \mu_3 \\ \mu_1 + \mu_3 + \mu_3 \end{pmatrix} \\
&= \begin{pmatrix} \mu_1 + \mu_3 \\ \mu_1 \\ \mu_1 + \mu_3 \end{pmatrix}.
\end{aligned}$$

Therefore $\lambda_2 = \mu_1 = \mu_3$. Writing $\mu = \mu_1 (= \mu_3)$, we have

$$\sigma\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ (ab)^{2^r}\mu \end{pmatrix}, \quad \sigma\begin{pmatrix} d^{-1} & 0 \\ c & d \end{pmatrix} = \begin{pmatrix} (cd)^{2^r}\mu \\ 0 \\ 0 \end{pmatrix}, \quad \sigma\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 \\ \mu \\ 0 \end{pmatrix}.$$

Suppose $c \neq 0$. Then

$$\begin{aligned}
\sigma\begin{pmatrix} a & b \\ c & d \end{pmatrix} &= \sigma\left(\begin{pmatrix} 1 & ac^{-1} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} c & d \\ 0 & c^{-1} \end{pmatrix}\right) \\
&= \sigma\begin{pmatrix} 1 & ac^{-1} \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & ac^{-1} \\ 0 & 1 \end{pmatrix} \cdot \left(\sigma\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot \sigma\begin{pmatrix} c & d \\ 0 & c^{-1} \end{pmatrix}\right) \\
&= \begin{pmatrix} 0 \\ 0 \\ (ac^{-1})^{2^r}\mu \end{pmatrix} + \begin{pmatrix} 1 & ac^{-1} \\ 0 & 1 \end{pmatrix} \cdot \left(\begin{pmatrix} 0 \\ \mu \\ 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ (cd)^{2^r}\mu \end{pmatrix}\right) \\
&= \begin{pmatrix} 0 \\ 0 \\ (ac^{-1})^{2^r}\mu \end{pmatrix} + \begin{pmatrix} 1 & ac^{-1} \\ 0 & 1 \end{pmatrix} \cdot \left(\begin{pmatrix} 0 \\ \mu \\ 0 \end{pmatrix} + \begin{pmatrix} (cd)^{2^r}\mu \\ 0 \\ 0 \end{pmatrix}\right) \\
&= \begin{pmatrix} 0 \\ 0 \\ (ac^{-1})^{2^r}\mu \end{pmatrix} + \begin{pmatrix} 1 & ac^{-1} \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} (cd)^{2^r}\mu \\ \mu \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 \\ 0 \\ (ac^{-1})^{2^r}\mu \end{pmatrix} + \begin{pmatrix} (cd)^{2^r}\mu \\ \mu + (ac^{-1})^{2^r}(cd)^{2^r}\mu \\ (ac^{-1})^{2^{r+1}}(cd)^{2^r}\mu \end{pmatrix} \\
&= \begin{pmatrix} (cd)^{2^r}\mu \\ \mu + (ac^{-1})^{2^r}(cd)^{2^r}\mu \\ (ac^{-1})^{2^r}\mu + (ac^{-1})^{2^{r+1}}(cd)^{2^r}\mu \end{pmatrix} \\
&= \begin{pmatrix} (cd)^{2^r}\mu \\ (1+ad)^{2^r}\mu \\ (1+ad)^{2^r}(ac^{-1})^{2^r}\mu \end{pmatrix} \\
&= \begin{pmatrix} (cd)^{2^r}\mu \\ (bc)^{2^r}\mu \\ (ab)^{2^r}\mu \end{pmatrix}.
\end{aligned} \tag{6.7}$$

Note that substituting $c = 0$ gives

$$\sigma \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ (ab)^{2^r}\mu \end{pmatrix},$$

substituting $b = 0$ gives

$$\sigma \begin{pmatrix} d^{-1} & 0 \\ c & d \end{pmatrix} = \begin{pmatrix} (cd)^{2^r}\mu \\ 0 \\ 0 \end{pmatrix},$$

and substituting $a = d = 0, b = c = 1$ gives

$$\sigma \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 \\ \mu \\ 0 \end{pmatrix},$$

so we take Equation 6.7 to be the general form of a 1-cocycle $\sigma : SL_2(k) \to V_\alpha$ such that $\sigma(y) = 1$ for all $y \in T_2(k)$.

We can show that σ as defined in Equation 6.7 is a well-defined 1-cocycle by verifying that Equations 6.2–6.4 hold for $r = \alpha, -\alpha$ but we leave the details to the reader.

We can now calculate the 1-cohomology by finding $\tau \in Z^1(SL_2(k), V_\alpha)_{\omega_r}$ such that $\tau = \sigma + \chi_{\mathbf{v}}^{SL_2(k)}$, for some $\mathbf{v} \in V_\alpha$. We may assume $\tau(y) = 1$ for all $y \in T_2(k)$, hence $\mathbf{v}$ is fixed by the action of $T_2(k)$.

To this end, let $\mathbf{v} = \begin{pmatrix} 0 \\ v \\ 0 \end{pmatrix}$. Since $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} 0 \\ v \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ v \\ 0 \end{pmatrix}$, $\chi_v^{SL_2(k)}$ is trivial. This shows that for each $\mu \in k$, $\psi(\sigma_\mu)$ is a distinct element in the 1-cohomology, where

$$\sigma_\mu \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} (cd)^{2^r}\mu \\ (bc)^{2^r}\mu \\ (ad)^{2^r}\mu \end{pmatrix}.$$

We now consider the action of $Z(L_\alpha)^\circ$, the connected centre of the Levi subgroup L_α. We have $Z(L_\alpha)^\circ = \langle \gamma^\vee(x) \mid x \in k \rangle$ where γ is a root in Φ such that $\langle \alpha, \gamma \rangle = 0$.

Since $\langle \alpha, \alpha+\beta \rangle = 0$, we may choose $\gamma = \alpha+\beta$. Therefore $Z(L_\alpha)^\circ = \langle (\alpha+\beta)^\vee(x) \mid x \in k \rangle$. Taking an element $\mathbf{s} = (\alpha+\beta)^\vee(s)$ of $Z(L_\alpha)^\circ$ we compute the action of $\mathbf{s}$ on the 1-cocycle

σ_μ as follows.

$$
\begin{aligned}
(\mathbf{s}\cdot\sigma_\mu)\begin{pmatrix} a & b \\ c & d \end{pmatrix} &= (\alpha+\beta)^\vee(s)\epsilon_\beta\left(\mu(cd)^{2^r}\right)\epsilon_{\alpha+\beta}\left(\mu(bc)^{2^r}\right)\epsilon_{2\alpha+\beta}\left(\mu(ab)^{2^r}\right)(\alpha+\beta)^\vee(s)^{-1} \\
&= \epsilon_\beta\left(s^{\langle\beta,\alpha+\beta\rangle}\mu(cd)^{2^r}\right)\epsilon_{\alpha+\beta}\left(s^{\langle\alpha+\beta,\alpha+\beta\rangle}\mu(bc)^{2^r}\right)\epsilon_{2\alpha+\beta}\left(s^{\langle 2\alpha+\beta,\alpha+\beta\rangle}\mu(ab)^{2^r}\right) \\
&= \begin{pmatrix} (s^2\mu)(cd)^{2^r} \\ (s^2\mu)(bc)^{2^r} \\ (s^2\mu)(ab)^{2^r} \end{pmatrix}.
\end{aligned}
$$

So we see that the infinitely many equivalence classes of 1-cocycles collapse to just two classes when we consider the action of $Z(L_\alpha)^\circ$, represented by the 1-cocycles σ_0 and σ_1. Hence by Theorem 4.19 there are only finitely many P-conjugacy classes of representations from K to P for each choice of ω.

6.2.2 $V = R_u(P_\beta)$

Let $V_\beta = R_u(P_\beta) = \langle U_\alpha, U_{\alpha+\beta}, U_{2\alpha+\beta}\rangle$.

Note that V is not abelian in general. The Group Law for V can be computed as follows. Writing $\mathbf{v}\in V_\beta$ as

$$
\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \epsilon_\alpha(v_1)\epsilon_{\alpha+\beta}(v_2)\epsilon_{2\alpha+\beta}(v_3),
$$

for $\mathbf{v},\mathbf{w}\in V_\beta$ we have

$$
\begin{aligned}
\mathbf{vw} &= \epsilon_\alpha(v_1)\epsilon_{\alpha+\beta}(v_2)\epsilon_{2\alpha+\beta}(v_3)\epsilon_\alpha(w_1)\epsilon_{\alpha+\beta}(w_2)\epsilon_{2\alpha+\beta}(w_3) \\
&= \epsilon_\alpha(v_1)\epsilon_{\alpha+\beta}(v_2)\epsilon_\alpha(w_1)\epsilon_{\alpha+\beta}(w_2)\epsilon_{2\alpha+\beta}(v_3)\epsilon_{2\alpha+\beta}(w_3) \\
&= \epsilon_\alpha(v_1)\epsilon_\alpha(w_1)\epsilon_{\alpha+\beta}(v_2)\epsilon_{2\alpha+\beta}(2v_2w_1)\epsilon_{\alpha+\beta}(w_2)\epsilon_{2\alpha+\beta}(v_3)\epsilon_{2\alpha+\beta}(w_3) \\
&= \epsilon_\alpha(v_1+w_1)\epsilon_{\alpha+\beta}(v_2+w_2)\epsilon_{2\alpha+\beta}(v_3+w_3+2v_2w_1) \\
&= \begin{pmatrix} v_1+w_1 \\ v_2+w_2 \\ v_3+w_3+2v_2w_1 \end{pmatrix}.
\end{aligned}
$$

We compute the following.

$$\begin{aligned}
\begin{pmatrix}1 & u\\ 0 & 1\end{pmatrix}\cdot\mathbf{v} &= \omega_r\begin{pmatrix}1 & u\\ 0 & 1\end{pmatrix}\mathbf{v}\left(\omega_r\begin{pmatrix}1 & u\\ 0 & 1\end{pmatrix}\right)^{-1}\\
&= \epsilon_\beta(u^{p^r})\epsilon_\alpha(v_1)\epsilon_{\alpha+\beta}(v_2)\epsilon_{2\alpha+\beta}(v_3)\epsilon_\beta(-u^{p^r})\\
&= \epsilon_\alpha(v_1)\epsilon_\beta(u^{p^r})\epsilon_{\alpha+\beta}(u^{p^r}v_1)\epsilon_{2\alpha+\beta}(u^{p^r}v_1^2)\epsilon_{\alpha+\beta}(v_2)\epsilon_{2\alpha+\beta}(v_3)\epsilon_\beta(-u^{p^r})\\
&= \epsilon_\alpha(v_1)\epsilon_\beta(u^{p^r})\epsilon_{\alpha+\beta}(v_2+u^{p^r}v_1)\epsilon_{2\alpha+\beta}(v_3+u^{p^r}v_1^2)\epsilon_\beta(-u^{p^r})\\
&= \epsilon_\alpha(v_1)\epsilon_{\alpha+\beta}(u^{p^r}v_1)\epsilon_{2\alpha+\beta}(u^{p^r}v_1^2)\epsilon_{\alpha+\beta}(v_2)\epsilon_{2\alpha+\beta}(v_3)\epsilon_\beta(u^{p^r})\epsilon_\beta(-u^{p^r})\\
&= \epsilon_\alpha(v_1)\epsilon_{\alpha+\beta}(v_2+u^{p^r}v_1)\epsilon_{2\alpha+\beta}(v_3+u^{p^r}v_1^2)\\
&= \begin{pmatrix}v_1\\ v_2+u^{p^r}v_1\\ v_3+u^{p^r}v_1^2\end{pmatrix}\\
\begin{pmatrix}t & 0\\ 0 & t^{-1}\end{pmatrix}\cdot\mathbf{v} &= \omega_r\begin{pmatrix}t & 0\\ 0 & t^{-1}\end{pmatrix}\mathbf{v}\left(\omega_r\begin{pmatrix}t & 0\\ 0 & t^{-1}\end{pmatrix}\right)^{-1}\\
&= \beta^\vee(t^{p^r})\epsilon_\alpha(v_1)\epsilon_{\alpha+\beta}(v_2)\epsilon_{2\alpha+\beta}(v_3)(\beta^\vee(t^{p^r}))^{-1}\\
&= \epsilon_\alpha\left(\alpha(\beta^\vee(t^{p^r}))v_1\right)\epsilon_{\alpha+\beta}\left((\alpha+\beta)(\beta^\vee(t^{p^r}))v_2\right)\epsilon_{2\alpha+\beta}\left((2\alpha+\beta)(\beta^\vee(t^{p^r}))v_3\right)\\
&= \epsilon_\alpha\left((t^{p^r})^{\langle\alpha,\beta\rangle}v_1\right)\epsilon_{\alpha+\beta}\left((t^{p^r})^{\langle\alpha+\beta,\beta\rangle}v_2\right)\epsilon_{2\alpha+\beta}\left((t^{p^r})^{\langle2\alpha+\beta,\beta\rangle}v_3\right)\\
&= \begin{pmatrix}t^{-p^r}v_1\\ t^{p^r}v_2\\ v_3\end{pmatrix}\\
\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\cdot\mathbf{v} &= \omega_r\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\mathbf{v}\left(\omega_r\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\right)^{-1}\\
&= n_\beta\epsilon_\alpha(v_1)\epsilon_{\alpha+\beta}(v_2)\epsilon_{2\alpha+\beta}(v_3)n_\beta^{-1}\\
&= n_\beta\epsilon_\alpha(v_1)n_\beta^{-1}n_\beta\epsilon_{\alpha+\beta}(v_2)n_\beta^{-1}n_\beta\epsilon_{2\alpha+\beta}(v_3)n_\beta^{-1}\\
&= \epsilon_{\alpha+\beta}(v_1)\epsilon_\alpha(-v_2)\epsilon_{2\alpha+\beta}(v_3)\\
&= \epsilon_\alpha(-v_2)\epsilon_{\alpha+\beta}(v_1)\epsilon_{2\alpha+\beta}(v_3-2v_1v_2)\\
&= \begin{pmatrix}-v_2\\ v_1\\ v_3-2v_1v_2\end{pmatrix}.
\end{aligned}$$

This is enough to determine the action of $SL_2(k)$ on V_β (we omit the details).

$$\begin{pmatrix}a & b\\ c & d\end{pmatrix}\cdot\mathbf{v} = \begin{pmatrix}c^{p^r}v_2+d^{p^r}v_1\\ a^{p^r}v_2+b^{p^r}v_1\\ v_3+(ac)^{p^r}v_2^2+(bd)^{p^r}v_1^2+2(bc)^{p^r}v_1v_2\end{pmatrix}.$$

Let $x \in H^1(SL_2(k), V_\beta)_{\omega_r}$. Then by Proposition 3.28 there exists $\sigma \in Z^1(SL_2(k), V_\beta)_{\omega_r}$ such that $\sigma(y) = 1$ for all $y \in T_2(k)$ and $\phi(\sigma) = x$. By Lemma 5.10 we can apply Lemma 5.11 to yield

$$\sigma\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} = \epsilon_{\alpha+\beta}(\mu_2(ab)^{2^{r-1}}),$$
$$\sigma\begin{pmatrix} d^{-1} & 0 \\ c & d \end{pmatrix} = \epsilon_{\alpha}(\mu_1(cd)^{2^{r-1}}) \quad \text{(Remark 5.12)},$$

for some $\mu_1, \mu_2 \in k$ if $p = 2$, and $\sigma(B_2(k)) = 1$ if $p > 2$. Hence by Lemma 3.27 $H^1(SL_2(k), V_\beta)_{\omega_r}$ is trivial if $p > 2$. Note that for σ to be a morphism we need $r > 0$.

Proceeding with $p = 2$ we see that V_β is abelian so we revert to additive notation.

By Equation 6.6 there exists $\lambda_3 \in k$ such that

$$\sigma\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \lambda_3 \end{pmatrix},$$

and by Equation 6.5

$$\begin{aligned}
\sigma\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} &= \sigma\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \cdot \left(\sigma\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \cdot \sigma\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\right) \\
&= \begin{pmatrix} 0 \\ \mu_2 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \cdot \left(\begin{pmatrix} \mu_1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ \mu_2 \\ 0 \end{pmatrix}\right) \\
&= \begin{pmatrix} 0 \\ \mu_2 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \cdot \left(\begin{pmatrix} \mu_1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} \mu_2 \\ \mu_2 \\ \mu_2^2 \end{pmatrix}\right) \\
&= \begin{pmatrix} 0 \\ \mu_2 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \mu_1 + \mu_2 \\ \mu_2 \\ \mu_2^2 \end{pmatrix} \\
&= \begin{pmatrix} 0 \\ \mu_2 \\ 0 \end{pmatrix} + \begin{pmatrix} \mu_1 + \mu_2 \\ \mu_1 \\ \mu_1^2 \end{pmatrix} \\
&= \begin{pmatrix} \mu_1 + \mu_2 \\ \mu_1 + \mu_2 \\ \mu_1^2 \end{pmatrix}.
\end{aligned}$$

Therefore $\mu_1 = \mu_2$ and $\lambda_3 = \mu_1^2 (= \mu_2^2)$. Writing $\mu = \mu_1 (= \mu_2)$, we have

$$\sigma \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} = \begin{pmatrix} 0 \\ \mu(ab)^{2^{r-1}} \\ 0 \end{pmatrix}, \quad \sigma \begin{pmatrix} d^{-1} & 0 \\ c & d \end{pmatrix} = \begin{pmatrix} \mu(cd)^{2^{r-1}} \\ 0 \\ 0 \end{pmatrix}, \quad \sigma \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \mu^2 \end{pmatrix}.$$

Suppose $c \neq 0$. Then

$$\begin{aligned}
\sigma \begin{pmatrix} a & b \\ c & d \end{pmatrix} &= \sigma \begin{pmatrix} 1 & ac^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} c & d \\ 0 & c^{-1} \end{pmatrix} \\
&= \sigma \begin{pmatrix} 1 & ac^{-1} \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & ac^{-1} \\ 0 & 1 \end{pmatrix} \cdot \left(\sigma \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot \sigma \begin{pmatrix} c & d \\ 0 & c^{-1} \end{pmatrix} \right) \\
&= \begin{pmatrix} 0 \\ \mu(ac^{-1})^{2^{r-1}} \\ 0 \end{pmatrix} + \begin{pmatrix} 1 & ac^{-1} \\ 0 & 1 \end{pmatrix} \cdot \left(\begin{pmatrix} 0 \\ 0 \\ \mu^2 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ \mu(cd)^{2^{r-1}} \\ 0 \end{pmatrix} \right) \\
&= \begin{pmatrix} 0 \\ \mu(ac^{-1})^{2^{r-1}} \\ 0 \end{pmatrix} + \begin{pmatrix} 1 & ac^{-1} \\ 0 & 1 \end{pmatrix} \cdot \left(\begin{pmatrix} 0 \\ 0 \\ \mu^2 \end{pmatrix} + \begin{pmatrix} \mu(cd)^{2^{r-1}} \\ 0 \\ 0 \end{pmatrix} \right) \\
&= \begin{pmatrix} 0 \\ \mu(ac^{-1})^{2^{r-1}} \\ 0 \end{pmatrix} + \begin{pmatrix} 1 & ac^{-1} \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \mu(cd)^{2^{r-1}} \\ 0 \\ \mu^2 \end{pmatrix} \\
&= \begin{pmatrix} 0 \\ \mu(ac^{-1})^{2^{r-1}} \\ 0 \end{pmatrix} + \begin{pmatrix} \mu(cd)^{2^{r-1}} \\ (ac^{-1})^{2^r} \mu(cd)^{2^{r-1}} \\ \mu^2 + (ac^{-1})^{2^r} \left(\mu(cd)^{2^{r-1}} \right)^2 \end{pmatrix} \\
&= \begin{pmatrix} \mu(cd)^{2^{r-1}} \\ \mu \left(ac^{-1} + a^2c^{-1}d \right)^{2^{r-1}} \\ \mu^2 \left(1 + ad \right)^{2^r} \end{pmatrix} \\
&= \begin{pmatrix} \mu(cd)^{2^{r-1}} \\ \mu \left(ab \right)^{2^{r-1}} \\ \mu^2 \left(bc \right)^{2^r} \end{pmatrix}.
\end{aligned} \tag{6.8}$$

Note that substituting $c = 0$ gives

$$\sigma \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} = \begin{pmatrix} 0 \\ \mu(ab)^{2^{r-1}} \\ 0 \end{pmatrix},$$

substituting $b = 0$ gives

$$\sigma \begin{pmatrix} d^{-1} & 0 \\ c & d \end{pmatrix} = \begin{pmatrix} \mu(cd)^{2^{r-1}} \\ 0 \\ 0 \end{pmatrix},$$

and substituting $a = d = 0, b = c = 1$ gives

$$\sigma \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \mu^2 \end{pmatrix},$$

so we take Equation 6.8 to be the general form of a 1-cocycle $\sigma : SL_2(k) \to V_\beta$ such that $\sigma(y) = 1$ for all $y \in T_2(k)$. We can now show that Equation 6.8 is a well-defined by verifying Equations 6.2–6.4, but as in the previous calculation we omit the details.

We calculate the 1-cohomology by calculating $\tau = \sigma + \chi_{\mathbf{v}}^{SL_2(k)}$ such that $\mathbf{v} \in V_\beta$ is fixed under the action of $T_2(k)$. To this end let $\mathbf{v} = \begin{pmatrix} 0 \\ 0 \\ v_3 \end{pmatrix}$.

$$\begin{aligned}
\tau \begin{pmatrix} a & b \\ c & d \end{pmatrix} &= \mathbf{v} + \sigma \begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \mathbf{v} \\
&= \begin{pmatrix} 0 \\ 0 \\ v_3 \end{pmatrix} + \begin{pmatrix} \mu(cd)^{2^{r-1}} \\ \mu\,(ab)^{2^{r-1}} \\ \mu^2\,(bc)^{2^r} \end{pmatrix} + \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ v_3 \end{pmatrix} \\
&= \begin{pmatrix} 0 \\ 0 \\ v_3 \end{pmatrix} + \begin{pmatrix} \mu(cd)^{2^{r-1}} \\ \mu\,(ab)^{2^{r-1}} \\ \mu^2\,(bc)^{2^r} \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ v_3 \end{pmatrix} \\
&= \begin{pmatrix} \mu(cd)^{2^{r-1}} \\ \mu\,(ab)^{2^{r-1}} \\ \mu^2\,(bc)^{2^r} \end{pmatrix}.
\end{aligned}$$

Therefore, for each μ in k we get a distinct element of the 1-cohomology $\psi(\sigma_\mu)$, where

$$\sigma_\mu \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \mu(cd)^{2^{r-1}} \\ \mu\,(ab)^{2^{r-1}} \\ \mu^2\,(bc)^{2^r} \end{pmatrix}.$$

But as before if we consider the action of $Z(L_\beta)$ on our 1-cocycles

$$\begin{aligned}(\mathbf{s}\cdot\sigma_\mu)\begin{pmatrix}a & b\\ c & d\end{pmatrix} &= (2\alpha+\beta)^\vee(s)\cdot\sigma_\mu\left(\begin{pmatrix}a & b\\ c & d\end{pmatrix}\right)\\ &= \begin{pmatrix}(s\mu)(cd)^{2^{r-1}}\\ (s\mu)\,(ab)^{2^{r-1}}\\ (s\mu)^2\,(bc)^{2^r}\end{pmatrix}.\end{aligned}$$

our infinitely equivalence classes collapse to just two classes, $\psi(\sigma_0), \psi(\sigma_1)$. Therefore, by Theorem 4.19 there are only finitely many P-conjugacy classes of representations from K to P for each choice of ω.

6.3 $G = G_2$

Let $G = G_2$. Fix a maximal torus, labeling the positive simple roots $\Delta = \{\alpha, \beta\}$ with β being the long root.

6.3.1 $V = R_u(P_\alpha)$

Let $V_\alpha = R_u(P_\alpha) = \langle U_\beta, U_{\alpha+\beta}, U_{2\alpha+\beta}, U_{3\alpha+\beta}, U_{3\alpha+2\beta}\rangle$. We write $v \in V_\alpha$ as a column vector,

$$\begin{pmatrix}v_1\\ v_2\\ v_3\\ v_4\\ v_5\end{pmatrix} = \epsilon_\beta(v_1)\epsilon_{\alpha+\beta}(v_2)\epsilon_{2\alpha+\beta}(v_3)\epsilon_{3\alpha+\beta}(v_4)\epsilon_{3\alpha+2\beta}(v_5),$$

so that we can clearly present the group law for V_α, derived from the commutation relations in [10, §33.5].

$$\begin{pmatrix}u_1\\ u_2\\ u_3\\ u_4\\ u_5\end{pmatrix}\times\begin{pmatrix}v_1\\ v_2\\ v_3\\ v_4\\ v_5\end{pmatrix} = \begin{pmatrix}u_1+v_1\\ u_2+v_2\\ u_3+v_3\\ u_4+v_4\\ u_5+v_5+3u_3v_2-u_4v_1\end{pmatrix}.$$

Let σ be in $Z^1(SL_2, V_\alpha)_{\omega_r}$ such that $\sigma\begin{pmatrix} t & 0 \\ 0 & t^{-1}\end{pmatrix} = 1$. By Proposition 5.1

$$\sigma\begin{pmatrix} 1 & u \\ 0 & 1\end{pmatrix} = \epsilon_{2\alpha+\beta}(x_3(u))\epsilon_{3\alpha+\beta}(x_4(u))$$

where the x_3, x_4 satisfy

$$x_3(t^2u) = t^{p^r}x_3(u) \tag{6.9}$$
$$x_4(t^2b) = t^{3p^r}x_4(u), \tag{6.10}$$

for all $t \in k^*, u \in k$.

Furthermore, since

$$\sigma\begin{pmatrix} 1 & u_1+u_2 \\ 0 & 1\end{pmatrix} = \left(\sigma\begin{pmatrix} 1 & u_1 \\ 0 & 1\end{pmatrix}\right)\begin{pmatrix} 1 & u_1 \\ 0 & 1\end{pmatrix}\cdot\sigma\begin{pmatrix} 1 & u_2 \\ 0 & 1\end{pmatrix},$$

we get

$$\begin{aligned}
&\epsilon_{2\alpha+\beta}\left(x_3(u_1+u_2)\right)\epsilon_{3\alpha+\beta}\left(x_4(u_1+u_2)\right)\\
&\quad= \epsilon_{2\alpha+\beta}\left(x_3(u_1)\right)\epsilon_{3\alpha+\beta}\left(x_4(u_1)\right)\epsilon_\alpha(u_1^{p^r})\epsilon_{2\alpha+\beta}\left(x_3(u_2)\right)\epsilon_{3\alpha+\beta}\left(x_4(u_2)\right)\epsilon_\alpha(-u_1^{p^r})\\
&\quad= \epsilon_{2\alpha+\beta}\left(x_3(u_1)\right)\epsilon_{3\alpha+\beta}\left(x_4(u_1)\right)\epsilon_\alpha(u_1^{p^r})\epsilon_{2\alpha+\beta}\left(x_3(u_2)\right)\epsilon_\alpha(-u_1^{p^r})\epsilon_{3\alpha+\beta}\left(x_4(u_2)\right)\\
&\quad= \epsilon_{2\alpha+\beta}\left(x_3(u_1)\right)\epsilon_{3\alpha+\beta}\left(x_4(u_1)\right)\epsilon_{2\alpha+\beta}\left(x_3(u_2)\right)\epsilon_{3\alpha+\beta}\left(x_4(u_2)-3u_1^{p^r}x_3(u_2)\right)\\
&\quad= \epsilon_{2\alpha+\beta}\left(x_3(u_1)+x_3(u_2)\right)\epsilon_{3\alpha+\beta}\left(x_4(u_1)+x_4(u_2)-3u_1^{p^r}x_3(u_2)\right)
\end{aligned}$$

We see that x_3 is an additive polynomial, so it is of the form

$$x_3(\lambda) = \sum_{i=0}^{m}\mu_i\lambda^{p^i} \quad ([10, §20.3, \text{Lemma A}]),$$

for some $\mu_i \in k, m \in \mathbb{N}$, and x_4 satisfies

$$x_4(\lambda_1+\lambda_2) = x_4(\lambda_1) + x_4(\lambda_2) - 3\lambda_2^{p^r}x_3(\lambda_2). \tag{6.11}$$

Suppose $x_3 \neq 0$, so there exists $j \geq 0$ such that $\mu_j \neq 0$. Then by Equation 6.9

$$\begin{aligned}
&\mu_j(t^2u)^{p^j} = t^{p^r}\mu_j u^{p^j}\\
&\Rightarrow t^{2p^j} = t^{p^r}\\
&\Rightarrow p = 2, j = r-1.
\end{aligned}$$

But then $x_3 = 0$, for

$$\begin{aligned} x_4(0) &= x_4(\lambda + \lambda) \\ &= x_4(\lambda) + x_4(\lambda) - 3\lambda^{2^r} x_3(\lambda) \quad \text{(Equation 6.11)} \\ &= 3\lambda^{2^r} x_3(\lambda), \end{aligned}$$

which implies that x_3 is constant, hence zero, as $\sigma(T_2(k)) = 1$.

Therefore $x_3 = 0$ in any case, and so by Equation 6.11, x_4 is an additive polynomial. Then it is of the form

$$x_4(\lambda) = \sum_{i=0}^{n} \nu_i \lambda^{p^r},$$

for some $\nu_i \in k, n \in \mathbb{N}$. If $x_4 \neq 0$ then some $\nu_j \neq 0$, and we get

$$\begin{aligned} &\nu_j (t^2 u)^{p^j} = t^{3p^r} \nu_j u^{p^j} \quad \text{(Equation 6.10)} \\ &\Rightarrow t^{2p^j} = t^{3p^r} \\ &\Rightarrow 2p^j = 3p^r, \end{aligned}$$

which implies that 2 divides p and 3 divides p, a contradiction. Hence $x_4 = 0$ and

$$\begin{aligned} \sigma \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} &= \sigma \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \quad \text{(Proposition 5.3)} \\ &= 1. \end{aligned}$$

Therefore $H^1(SL_2(k), V_\alpha)_{\omega_r}$ is trivial by Lemma 3.27.

The purpose of this calculation was to show that although the hypotheses of Lemma 5.10 fail (Remark 5.5) we still (trivially) obtain the same result as the conclusion of Lemma 5.11; that is, the equation for $\sigma(B_2(k))$.

6.4 $G = C_3$

Let $G = C_3$. Fix a maximal torus, labeling the positive simple roots $\Delta = \{\alpha, \beta, \gamma\}$ with γ being the long root and connected to β.

6.4.1 $V = R_u(P_\alpha)$

Let $V_\alpha = R_u(P_\alpha) = \langle U_\beta, U_\gamma, U_{\alpha+\beta}, U_{\beta+\gamma}, U_{\alpha+\beta+\gamma}, U_{2\beta+\gamma}, U_{\alpha+2\beta+\gamma}, U_{2\alpha+2\beta+\gamma}\rangle$. We will write v in V_α as a column vector,

$$\begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \\ v_7 \\ v_8 \end{pmatrix} = \epsilon_\beta(v_1)\epsilon_\gamma(v_2)\epsilon_{\alpha+\beta}(v_3)\epsilon_{\beta+\gamma}(v_4)\epsilon_{\alpha+\beta+\gamma}(v_5)\epsilon_{2\beta+\gamma}(v_6)\epsilon_{\alpha+2\beta+\gamma}(v_7)\epsilon_{2\alpha+2\beta+\gamma}(v_8),$$

so that we can write the group law for V_α ([10, §33.3, §33.4]) as

$$\begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \\ u_8 \end{pmatrix} \times \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \\ v_7 \\ v_8 \end{pmatrix} = \begin{pmatrix} u_1 + v_1 \\ u_2 + v_2 \\ u_3 + v_3 \\ u_4 + v_4 + u_2 v_1 \\ u_5 + v_5 - u_3 v_2 \\ u_6 + v_6 + u_2 v_1^2 + 2u_4 v_1 \\ u_7 + v_7 + u_2 u_3 v_1 + u_2 v_1 v_3 + u_5 v_1 + u_4 v_3 \\ u_8 + v_8 - u_3^2 v_2 - 2u_3 v_2 v_3 + 2u_5 v_3 \end{pmatrix}.$$

Let σ be in $Z^1(SL_2, V_\alpha)$ such that $\sigma\begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}$. By Proposition 5.1

$$\sigma\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} = \epsilon_{\alpha+\beta}\left(x_3(u)\right)\epsilon_{\alpha+\beta+\gamma}\left(x_5(u)\right)\epsilon_{2\alpha+2\beta+\gamma}\left(x_8(u)\right)$$

where x_3, x_5, x_8 satisfy

$$x_3(t^2 u) = t^{p^r} x_3(u) \tag{6.12}$$

$$x_5(t^2 u) = t^{p^r} x_5(u) \tag{6.13}$$

$$x_8(t^2 u) = t^{2p^r} x_8(u), \tag{6.14}$$

Since $\alpha+\delta \notin \Phi$ for $\delta \in \{\alpha+\beta, \alpha+\beta+\gamma, 2\alpha+2\beta+\gamma\}$, $\sigma \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}$ is unchanged under the action of $\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$ for any $u \in k$. Then

$$\begin{aligned}
\sigma \begin{pmatrix} 1 & u_1+u_2 \\ 0 & 1 \end{pmatrix} &= \left(\sigma \begin{pmatrix} 1 & u_1 \\ 0 & 1 \end{pmatrix}\right) \left(\begin{pmatrix} 1 & u_1 \\ 0 & 1 \end{pmatrix} \cdot \sigma \begin{pmatrix} 1 & u_2 \\ 0 & 1 \end{pmatrix}\right) \\
&= \sigma \begin{pmatrix} 1 & u_1 \\ 0 & 1 \end{pmatrix} \sigma \begin{pmatrix} 1 & u_2 \\ 0 & 1 \end{pmatrix}.
\end{aligned}$$

Therefore

$$\begin{aligned}
&\epsilon_{\alpha+\beta}\left(x_3(u_1+u_2)\right)\epsilon_{\alpha+\beta+\gamma}\left(x_5(u_1+u_2)\right)\epsilon_{2\alpha+2\beta+\gamma}\left(x_8(u_1+u_2)\right) \\
&= \epsilon_{\alpha+\beta}\left(x_3(u_1)\right)\epsilon_{\alpha+\beta+\gamma}\left(x_5(u_1)\right)\epsilon_{2\alpha+2\beta+\gamma}\left(x_8(u_1)\right)\epsilon_{\alpha+\beta}\left(x_3(u_2)\right) \\
&\qquad \epsilon_{\alpha+\beta+\gamma}\left(x_5(u_2)\right)\epsilon_{2\alpha+2\beta+\gamma}\left(x_8(u_2)\right) \\
&= \epsilon_{\alpha+\beta}\left(x_3(u_1)\right)\epsilon_{\alpha+\beta+\gamma}\left(x_5(u_1)\right)\epsilon_{\alpha+\beta}\left(x_3(u_2)\right)\epsilon_{2\alpha+2\beta+\gamma}\left(x_8(u_1)\right) \\
&\qquad \epsilon_{\alpha+\beta+\gamma}\left(x_5(u_2)\right)\epsilon_{2\alpha+2\beta+\gamma}\left(x_8(u_2)\right) \\
&= \epsilon_{\alpha+\beta}\left(x_3(u_1)\right)\epsilon_{\alpha+\beta}\left(x_3(u_2)\right)\epsilon_{\alpha+\beta+\gamma}\left(x_5(u_1)\right)\epsilon_{2\alpha+2\beta+\gamma}\left(x_8(u_1)+2x_3(u_2)x_5(u_1)\right) \\
&\qquad \epsilon_{\alpha+\beta+\gamma}\left(x_5(u_2)\right)\epsilon_{2\alpha+2\beta+\gamma}\left(x_8(u_2)\right) \\
&= \epsilon_{\alpha+\beta}\left(x_3(u_1)+x_3(u_2)\right)\epsilon_{\alpha+\beta+\gamma}\left(x_5(u_1)+x_5(u_2)\right) \\
&\qquad \epsilon_{2\alpha+2\beta+\gamma}\left(x_8(u_1)+x_8(u_2)+2x_3(u_2)x_5(u_1)\right)
\end{aligned}$$

We see that x_3, x_5 are additive polynomials, so by [10, §20.3, Lemma A] they are of the form

$$\begin{aligned}
x_3(\lambda) &= \sum_{i=0}^{m} \mu_i \lambda^{p^r} \\
x_5(\lambda) &= \sum_{i=0}^{n} \nu_i \lambda^{p^r},
\end{aligned}$$

for some $\mu_i, \nu_i \in k, m, n \in \mathbb{N}$, while x_8 is of the form

$$x_8(\lambda_1+\lambda_2) = x_8(\lambda_1)+x_8(\lambda_2)+2x_3(\lambda_2)x_5(\lambda_1). \tag{6.15}$$

Suppose $x_3 \neq 0$. Then there exists $j \geq 0$ such that $\mu_j \neq 0$, so by Equation 6.12

$$\begin{aligned}
&\mu_j(t^2u)^{p^j} = t^{p^r}\mu_j u^{p^j} \\
&\Rightarrow t^{2p^j} = t^{p^r} \\
&\Rightarrow 2p^j = p^r.
\end{aligned}$$

Therefore $p = 2$, $j = r-1$ and $x_3(\lambda) = \mu\lambda^{2^{r-1}}$ for all $\lambda \in k$, for some fixed $\mu \in k$. The same argument shows that if $x_5 \neq 0$ then $p = 2$ and $x_5(\lambda) = \nu\lambda^{2^{r-1}}$ for all $\lambda \in k$, for some fixed $\nu \in k$.

Consider Equation 6.15. If x_3 or x_4 is nonzero then $p = 2$, so $2x_3(\lambda_2)x_5(\lambda_1) = 0$. Conversely $2x_3(\lambda_2)x_5(\lambda_1) = 0$ if either of x_3 or x_5 is zero. Hence in either case x_8 is an additive polynomial, so by [10, §20.3, Lemma A] x_8 is of the form

$$x_8(\lambda_1 + \lambda_2) = x_8(\lambda_1) + x_8(\lambda_2).$$

Moreover, in either case $\sigma(U_2(k))$ lies in an abelian subgroup of V_α. Hence Lemma 5.11 applies. The two cases are

$$\begin{aligned}
p = 2: \qquad & x_3(\lambda) = \mu\lambda^{2^{r-1}} \\
& x_5(\lambda) = \nu\lambda^{2^{r-1}} \\
& x_8(\lambda) = \omega\lambda^{2^r}, \\
p > 2: \qquad & x_3 = x_5 = 0 \\
& x_8(\lambda) = \omega\lambda^{p^r}.
\end{aligned}$$

Note that in the first case we require $r > 0$.

The point of this example calculation was to show that we could eventually apply Lemma 5.11 even though Proposition 5.7 does not apply, so we could not use Lemma 5.10. This provides some evidence for Conjecture 5.14.

6.5 $G = B_4$

Let $G = B_4$ and let $\mathrm{char}(k) = 2$. We label the positive simple roots $\Delta = \{\alpha, \beta, \gamma, \delta\}$, where γ is the long root connected to the short root δ, β is connected to γ, and α is connected to β. Let $P_{\gamma,\delta}$ be the rank 2 parabolic subgroup of G determined by the simple roots $\{\gamma, \delta\}$. Note that this is the only example in which we consider representations into a rank 2 parabolic. Let $V = R_u(P_{\gamma,\delta}) = \langle U_\phi \,|\, \phi \in \Phi^+, \phi \neq \gamma, \delta, \gamma+\delta, \gamma+2\delta\}\rangle$. We will write

$$\mathbf{v} = \epsilon_\alpha(v_1)\epsilon_\beta(v_2)\epsilon_{\alpha+\beta}(v_3)\epsilon_{\beta+\gamma}(v_4)\epsilon_{\alpha+\beta+\gamma}(v_5)\epsilon_{\beta+\gamma+\delta}(v_6)\epsilon_{\alpha+\beta+\gamma+\delta}(v_7)\epsilon_{\beta+\gamma+2\delta}(v_8)$$
$$\epsilon_{\alpha+\beta+\gamma+2\delta}(v_9)\epsilon_{\beta+2\gamma+2\delta}(v_{10})\epsilon_{\alpha+\beta+2\gamma+2\delta}(v_{11})\epsilon_{\alpha+2\beta+2\gamma+2\delta}(v_{12}) \in V$$

as a column vector:

$$\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \\ v_7 \\ v_8 \\ v_9 \\ v_{10} \\ v_{11} \\ v_{12} \end{pmatrix}.$$

The Group Law on V is

$$\mathbf{uv} = \mathbf{u} + \mathbf{v} + \begin{pmatrix} 0 \\ 0 \\ u_2v_1 \\ 0 \\ u_4v_1 \\ 0 \\ u_6v_1 \\ 0 \\ u_8v_1 \\ 0 \\ u_{10}v_1 \\ u_{10}v_1v_2 + u_8v_1v_4 + u_6^2v_1 + u_{11}v_2 + u_{10}v_3 + u_9v_4 + u_8v_5 \end{pmatrix}.$$

For integers $r, s \geq 0$ we have a homomorphism $\omega_{r,s} : SL_2 \to \widetilde{A}_1\widetilde{A}_1 < L_{\{\gamma,\delta\}}$ defined by

$$\omega_{r,s}\begin{pmatrix}1 & u\\ 0 & 1\end{pmatrix} = \epsilon_\delta(u^{2^r}) \cdot \epsilon_{\gamma+\delta}(u^{2^s})$$

$$\omega_{r,s}\begin{pmatrix}t & 0\\ 0 & t^{-1}\end{pmatrix} = \delta^\vee(t^{2^r}) \cdot (\gamma+\delta)^\vee(t^{2^s})$$

$$\omega_{r,s}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} = n_\delta \cdot n_{\gamma+\delta}$$

from which we obtain an action of SL_2 on V:

$$\begin{pmatrix}a & b\\ c & d\end{pmatrix} \cdot \mathbf{v} = \begin{pmatrix} v_1 \\ c^{2^{s+1}}v_{10} + d^{2^{s+1}}v_2 \\ c^{2^{s+1}}v_{11} + d^{2^{s+1}}v_3 \\ c^{2^{r+1}}v_8 + d^{2^{r+1}}v_4 \\ c^{2^{r+1}}v_9 + d^{2^{r+1}}v_5 \\ v_6 + (bd)^{2^r}v_4 + (bd)^{2^s}v_2 + (ac)^{2^r}v_8 + (ac)^{2^s}v_{10} \\ v_7 + (bd)^{2^r}v_5 + (bd)^{2^s}v_3 + (ac)^{2^r}v_9 + (ac)^{2^s}v_{11} \\ a^{2^{r+1}}v_8 + b^{2^{r+1}}v_4 \\ a^{2^{r+1}}v_9 + b^{2^{r+1}}v_5 \\ a^{2^{s+1}}v_{10} + b^{2^{s+1}}v_2 \\ a^{2^{s+1}}v_{11} + b^{2^{s+1}}v_3 \\ v_{12} + (bd)^{2^{r+1}}v_4v_5 + (bd)^{2^{s+1}}v_2v_3 + (bc)^{2^{r+1}}(v_4v_9 + v_5v_8) \\ + (bc)^{2^{s+1}}(v_2v_{11} + v_3v_{10}) + (ac)^{2^{r+1}}(v_8v_9) + (ac)^{2^{s+1}}(v_{10}v_{11}) \end{pmatrix}$$

Now let σ be a 1-cocycle from SL_2 to V such that for all t in k^*

$$\sigma\begin{pmatrix}t & 0\\ 0 & t^{-1}\end{pmatrix} = \begin{pmatrix}0\\ \vdots\\ 0\end{pmatrix}.$$

Since σ is a morphism of varieties, each component of $\sigma\begin{pmatrix}1 & u\\ 0 & 1\end{pmatrix}$ should be a polynomial function of u, so we let

$$\sigma\begin{pmatrix}1 & u\\ 0 & 1\end{pmatrix} = \begin{pmatrix}p_1(u)\\ \vdots\\ p_{12}(u)\end{pmatrix},$$

where each p_i $(1 \leq i \leq 12)$ is as required. Applying σ to the identity

$$\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \begin{pmatrix} t^{-1} & 0 \\ 0 & t \end{pmatrix} = \begin{pmatrix} 1 & t^2u \\ 0 & 1 \end{pmatrix},$$

gives rise to the following equations

$$p_i(t^2u) = \begin{cases} p_i(u), & i = 1, 6, 7, 12 \\ t^{-2^{r+1}} p_i(u), & i = 4, 5 \\ t^{-2^{s+1}} p_i(u), & i = 2, 3 \\ t^{2^{r+1}} p_i(u), & i = 8, 9 \\ t^{2^{s+1}} p_i(u), & i = 10, 11 \end{cases} \tag{6.16}$$

It is clear that for $i = 1, 6, 7, 12$ the polynomials p_i must be constant-valued, say λ_i for some fixed λ_i in k (resp). Furthermore, since $p_i(t^2u)$ involves only non-negative powers of t, p_i must be the zero polynomial for $i = 2, 3, 4, 5$. Now consider the identity

$$\begin{pmatrix} 1 & u_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & u_2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & u_1 + u_2 \\ 0 & 1 \end{pmatrix}.$$

Applying σ to both sides yields

$$\begin{aligned}
p_1(u_1 + u_2) &= p_1(u_1) + p_1(u_2) \\
p_6(u_1 + u_2) &= p_6(u_1) + p_6(u_2) \\
p_7(u_1 + u_2) &= p_7(u_1) + p_7(u_2) + p_6(u_1)p_1(u_2) \\
p_8(u_1 + u_2) &= p_8(u_1) + p_8(u_2) \\
p_9(u_1 + u_2) &= p_9(u_1) + p_9(u_2) + p_8(u_1)p_1(u_2) \\
p_{10}(u_1 + u_2) &= p_{10}(u_1) + p_{10}(u_2) \\
p_{11}(u_1 + u_2) &= p_{11}(u_1) + p_{11}(u_2) + p_{10}(u_1)p_1(u_2) \\
p_{12}(u_1 + u_2) &= p_{12}(u_1) + p_{12}(u_2) + (p_6(u_1))^2\, p_1(u_2).
\end{aligned}$$

Now we see that the constant polynomials p_1, p_6, p_7, p_{12} must in fact be the zero polynomial and the remaining polynomials must be homomorphisms from $k \to k$. That is for some w_j, x_j, y_j, z_j in k and all u in k

$$\begin{aligned} p_8(u) &= \sum_{j=0}^{N} w_j u^{2^j} \\ p_9(u) &= \sum_{j=0}^{N} x_j u^{2^j} \\ p_{10}(u) &= \sum_{j=0}^{N} y_j u^{2^j} \\ p_{11}(u) &= \sum_{j=0}^{N} z_j u^{2^j}, \end{aligned}$$

If σ is not the trivial 1-cocycle then one of the polynomials above is not the zero polynomial. Suppose for instance that p_8 is not the zero polynomial, so that $w_l \neq 0$ for some index $l \geq 0$. Then by Equation 6.16

$$\begin{aligned} \sum_{j=0}^{N} w_j (t^2 u)^{2^j} &= t^{2^{r+1}} \sum_{j=0}^{N} w_j u^{2^j} \\ \Rightarrow \quad w_l (t^2 u)^{2^l} &= t^{2^{r+1}} w_l u^{2^l} \\ \Rightarrow \quad l &= r. \end{aligned}$$

Similarly, for the polynomials $p_9, p_{10}, , p_{11}$, we get that

$$\begin{aligned} p_8(u) = wu^{2^r}, &\quad p_9(u) = xu^{2^r}, \\ p_{10}(u) = yu^{2^s}, &\quad p_{11}(u) = zu^{2^s}, \end{aligned}$$

for some w, x, y, z in k.

So, we have

$$\sigma\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} = \sigma\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}\left[\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}\cdot\sigma\begin{pmatrix} 1 & a^{-1}b \\ 0 & 1 \end{pmatrix}\right]$$

$$= \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ w(ab)^{2^{r+1}} \\ x(ab)^{2^{r+1}} \\ y(ab)^{2^{s+1}} \\ z(ab)^{2^{s+1}} \\ 0 \end{pmatrix}.$$

We apply the same argument using the fact that each component of $\sigma\begin{pmatrix} 1 & 0 \\ u & 1 \end{pmatrix}$ is a polynomial function, say $p'_i(u)$ for all u in k, to get

$$\sigma\begin{pmatrix} d^{-1} & 0 \\ c & d \end{pmatrix} = \begin{pmatrix} 0 \\ y'(cd)^{2^s} \\ z'(cd)^{2^s} \\ w'(cd)^{2^r} \\ x'(cd)^{2^r} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

for some w', x', y', z' in k.

From this we deduce that

$$
\begin{aligned}
\sigma\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} &= \sigma\left(\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix}\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}\right)\\
&= \sigma\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}\left[\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}\cdot\sigma\left(\begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix}\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}\right)\right]\\
&= \sigma\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}\left[\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}\cdot\left(\sigma\begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix}\left[\begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix}\cdot\sigma\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}\right]\right)\right]\\
&= \begin{pmatrix}0\\ y+y'\\ z+z'\\ w+w'\\ x+x'\\ w'+y'\\ x'+z'\\ w+w'\\ x+x'\\ y+y'\\ z+z'\\ w'x'+y'z'\end{pmatrix}.
\end{aligned}
$$

Furthermore, since $\sigma\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$ is fixed under the action of $\begin{pmatrix}t & 0\\ 0 & t^{-1}\end{pmatrix}$, we have

$$
\sigma\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} = \begin{pmatrix}n_1\\ 0\\ 0\\ 0\\ 0\\ n_6\\ n_7\\ 0\\ 0\\ 0\\ 0\\ n_{12}\end{pmatrix},
$$

for some n_1, n_6, n_7, n_{12} in k. So in fact

$$\begin{aligned} w' &= w \\ x' &= x \\ y' &= y \\ z' &= z \\ n_1 &= 0 \\ n_6 &= w + y \\ n_7 &= x + z \\ n_{12} &= wx + yz. \end{aligned}$$

Consider $\sigma \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. If $c = 0$ then we already have

$$\sigma \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} = \sigma \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \left[\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \cdot \sigma \begin{pmatrix} 1 & a^{-1}b \\ 0 & 1 \end{pmatrix} \right]$$
$$= \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ w(ab)^{2^{r+1}} \\ x(ab)^{2^{r+1}} \\ y(ab)^{2^{s+1}} \\ z(ab)^{2^{s+1}} \\ 0 \end{pmatrix}.$$

Otherwise, $c \neq 0$ and we can write

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & ac^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} c & d \\ 0 & c^{-1} \end{pmatrix},$$

and so

$$\begin{aligned}
\sigma\begin{pmatrix} a & b \\ c & d \end{pmatrix} &= \sigma\left(\begin{pmatrix} 1 & ac^{-1} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} c & d \\ 0 & c^{-1} \end{pmatrix}\right) \\
&= \sigma\begin{pmatrix} 1 & ac^{-1} \\ 0 & 1 \end{pmatrix}\left[\begin{pmatrix} 1 & ac^{-1} \\ 0 & 1 \end{pmatrix}\cdot\sigma\left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} c & d \\ 0 & c^{-1} \end{pmatrix}\right)\right] \\
&= \sigma\begin{pmatrix} 1 & ac^{-1} \\ 0 & 1 \end{pmatrix}\left[\begin{pmatrix} 1 & ac^{-1} \\ 0 & 1 \end{pmatrix}\cdot\left(\sigma\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\left[\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\cdot\sigma\begin{pmatrix} c & d \\ 0 & c^{-1} \end{pmatrix}\right]\right)\right] \\
&= \begin{pmatrix} 0 \\ y(cd)^{2^s} \\ z(cd)^{2^s} \\ w(cd)^{2^r} \\ x(cd)^{2^r} \\ n_6 + w(ad)^{2^r} + y(ad)^{2^s} \\ n_7 + x(ad)^{2^r} + z(ad)^{2^s} \\ w(ab)^{2^r} \\ x(ab)^{2^r} \\ y(ab)^{2^s} \\ z(ab)^{2^r} \\ n_{12} + wx(ad)^{2^{r+1}} + yz(ad)^{2^{s+1}} \end{pmatrix} \\
&= \begin{pmatrix} 0 \\ y(cd)^{2^s} \\ z(cd)^{2^s} \\ w(cd)^{2^r} \\ x(cd)^{2^r} \\ w(bc)^{2^r} + y(bc)^{2^s} \\ x(bc)^{2^r} + z(bc)^{2^s} \\ w(ab)^{2^r} \\ x(ab)^{2^r} \\ y(ab)^{2^s} \\ z(ab)^{2^r} \\ wx(bc)^{2^{r+1}} + yz(bc)^{2^{s+1}} \end{pmatrix}.
\end{aligned}$$

We see that in any case

$$\sigma\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 \\ y(cd)^{2^s} \\ z(cd)^{2^s} \\ w(cd)^{2^r} \\ x(cd)^{2^r} \\ w(bc)^{2^r} + y(bc)^{2^s} \\ x(bc)^{2^r} + z(bc)^{2^s} \\ w(ab)^{2^r} \\ x(ab)^{2^r} \\ y(ab)^{2^s} \\ z(ab)^{2^r} \\ wx(bc)^{2^{r+1}} + yz(bc)^{2^{s+1}} \end{pmatrix}.$$

This is our candidate for a general 1-cocycle on $SL_2(k)$. It turns out that σ is a well-defined 1-cocycle on $SL_2(k)$ by verifying Equations 6.2–6.4 but we omit the details here.

Given the general form of a 1-cocycle we can now calculate the 1-cohomology. We let $\tau \in \psi(\sigma)$, assuming that $\tau(T_2(k)) = 1$. Then there exists $\mathbf{v}$ in V that is fixed under the action of $\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}$, such that $\tau(x) = \mathbf{v}\sigma(x)(x \cdot \mathbf{v}^{-1})$, for all $x \in SL_2(k)$.

We have

$$\tau\begin{pmatrix}a & b\\ c & d\end{pmatrix} = \mathbf{v}\sigma\begin{pmatrix}a & b\\ c & d\end{pmatrix}\left[\begin{pmatrix}a & b\\ c & d\end{pmatrix}\cdot\mathbf{v}^{-1}\right]$$

$$= \begin{pmatrix}v_1\\0\\0\\0\\0\\v_6\\v_7\\0\\0\\0\\0\\v_{12}\end{pmatrix}\sigma\begin{pmatrix}a & b\\ c & d\end{pmatrix}\left[\begin{pmatrix}a & b\\ c & d\end{pmatrix}\cdot\begin{pmatrix}v_1\\0\\0\\0\\0\\v_6\\v_7+v_1v_6\\0\\0\\0\\0\\v_{12}+v_1v_6^2\end{pmatrix}\right]$$

$$= \begin{pmatrix}0\\ y(cd)^{2^s}\\ (z+yv_1)(cd)^{2^s}\\ w(cd)^{2^r}\\ (x+wv_1)(cd)^{2^r}\\ w(bc)^{2^r}+y(bc)^{2^s}\\ (x+wv_1)(bc)^{2^r}+(z+yv_1)(bc)^{2^s}\\ w(ab)^{2^r}\\ (x+wv_1)(ab)^{2^r}\\ y(ab)^{2^s}\\ (z+yv_1)(ab)^{2^r}\\ w(x+wv_1)(bc)^{2^{r+1}}+y(z+yv_1)(bc)^{2^{s+1}}\end{pmatrix}.$$

We can denote this relationship by

$$(w,x,y,z)\sim(w,x+\lambda w,y,z+\lambda y),$$

where the 4-tuple (w, x, y, z) represents the 1-cocycle

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} 0 \\ y(cd)^{2^s} \\ z(cd)^{2^s} \\ w(cd)^{2^r} \\ x(cd)^{2^r} \\ w(bc)^{2^r} + y(bc)^{2^s} \\ x(bc)^{2^r} + z(bc)^{2^s} \\ w(ab)^{2^r} \\ x(ab)^{2^r} \\ y(ab)^{2^s} \\ z(ab)^{2^r} \\ wx(bc)^{2^{r+1}} + yz(bc)^{2^{s+1}} \end{pmatrix}.$$

That is, $\psi((w, x, y, z)) = \psi((w', x', y', z'))$ if and only if $w' = w, x' = x + \lambda w, y' = y, z' = z + \lambda y$, for some $\lambda \in k$.

We find infinitely many equivalence classes of 1-cocycles. For instance, for each $x, z \in k$, $\psi((0, x, 0, z))$ is a distinct element of the 1-cohomology.

Now we consider the action of $Z(L_{\{\gamma,\delta\}})^\circ$ on the 1-cohomology. An element $\mathbf{s} = \alpha^\vee(t)(\beta+\gamma+\delta)^\vee(u) \in Z(L_{\{\gamma,\delta\}})^\circ$ acts on the 1-cocycle σ by

$$(\mathbf{s} \cdot \sigma) \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 \\ t^{-1}u^2y(cd)^{2^s} \\ tz(cd)^{2^s} \\ t^{-1}u^2w(cd)^{2^r} \\ tx(cd)^{2^r} \\ t^{-1}u^2(w(bc)^{2^r} + y(bc)^{2^s}) \\ t(x(bc)^{2^r} + z(bc)^{2^s}) \\ t^{-1}u^2w(ab)^{2^r} \\ tx(ab)^{2^r} \\ t^{-1}u^2y(ab)^{2^s} \\ tz(ab)^{2^r} \\ u^2(wx(bc)^{2^{r+1}} + yz(bc)^{2^{s+1}}) \end{pmatrix}.$$

In this case we can find a 1-parameter family of 1-cocycles, for example $(0, 1, 0, z), z \in k$, whose projections to $H^1(SL_2(k), V)_{\omega_{r,s}}/Z(L_{\gamma,\delta})^\circ$ are distinct equivalence classes. This shows that by Theorem 4.19 there exist infinitely many P-conjugacy classes of representations from K to P for the particular choice of $\omega_{r,s}$.

Hence by Lemma 4.4 we have infinitely many G-conjugacy classes of representations from K to G.

Chapter 7

Future work

One direction to pursue would be the further investigation of the B_4 calculation in Section 6.5. The discovery of infinitely many embeddings of $SL_2(k)$ in B_4 is encouraging as a potential counterexample to the algebraic version of Külshammer's second question.

It would also be interesting to prove or disprove Conjecture 5.14, perhaps for arbitrary parabolics P. This would provide a formula for candidate 1-cocycles from which we can use to calculate the nonabelian 1-cohomology by direct computation or computer program, following the examples of Chapter 6.

We would also like to fully explore the consequences of Lemma 5.10 with regards to the restriction of 1-cohomologies

$$H^1(SL_2(k), V_\alpha)_{\omega_r} \to H^1(U_2(k), V_\alpha)_{\omega_r},$$

where we may be able to use the fact that σ lies in an abelian subgroup of V_α to show that the map is injective (cf. Example 3.2).

A further course of interest is the thorough examination of Cram's non-reductive counterexample to Külshammer's second question ([1, Appendix]). We may be able to exploit certain properties of Cram's counterexample to construct a reductive counterexample, or alternatively, provide some insight towards a proof that there is no reductive counterexample.

Appendix A

Auxiliary Calculations

A.1

We show that $U_2(\mathbb{F}_{p^r})$ is a Sylow p-subgroup of $SL_2(\mathbb{F}_{p^r})$.

The group $GL_2(\mathbb{F}_{p^r})$ has order $(p^{2r}-1)(p^{2r}-p^r)$ since there are $p^{2r}-1$ choices of vectors for the first column (all choices excluding the zero vector), and $p^{2r}-p^r$ choices of vectors for the second column (all choices excluding multiples of the first vector). The determinant is a homomorphism of groups

$$\det : GL_2(\mathbb{F}_{p^r}) \to \mathbb{F}^*_{p^r},$$

with kernel $SL_2(\mathbb{F}_{p^r})$. Therefore, by the First Homomorphism Theorem for groups

$$GL_2(\mathbb{F}_{p^r}) \,/\, SL_2(\mathbb{F}_{p^r}) \simeq \det(GL_2(\mathbb{F}_{p^r})) = \mathbb{F}^*_{p^r},$$

and so

$$\begin{aligned} |SL_2(\mathbb{F}_{p^r})| &= |GL_2(\mathbb{F}_{p^r})| \,/\, |\mathbb{F}^*_{p^r}| \\ &= (p^{2r}-1)(p^{2r}-p^r) \,/\, (p^r-1) \\ &= p^r(p^{2r}-1). \end{aligned}$$

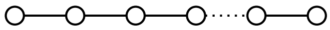
Since $|U(\mathbb{F}_{p^r})| = p^r$, $U(\mathbb{F}_{p^r})$ is a Sylow p-subgroup of $SL_2(\mathbb{F}_{p^r})$. $\square$

Appendix B

Root System Diagrams

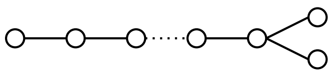
B.1 Dynkin Diagrams

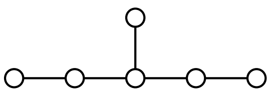
Short roots are identified by filled circles.

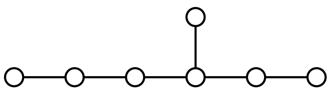
A_n : 

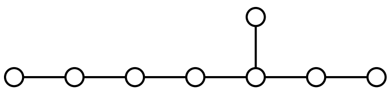
B_n : 

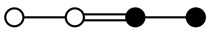
C_n : 

D_n : 

E_6 : 

E_7 : 

E_8 : 

F_4 : 

G_2 : 

B.2 Cartan Matrix

The matrix of Cartan integers for root systems of rank 2, defined by

$$\begin{pmatrix} \langle \alpha, \alpha \rangle & \langle \alpha, \beta \rangle \\ \langle \beta, \alpha \rangle & \langle \beta, \beta \rangle \end{pmatrix}.$$

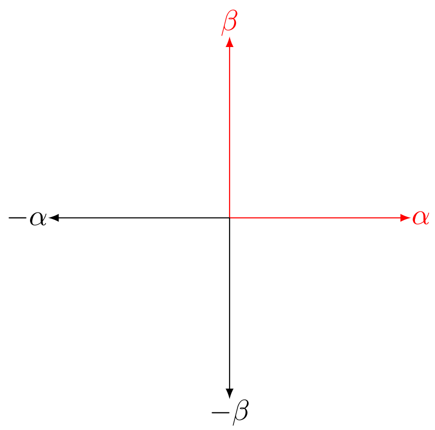
If the roots are of different length then α is the shorter root.

$$A_1 \times A_1 : \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \qquad B_2 : \begin{pmatrix} 2 & -1 \\ -2 & 2 \end{pmatrix}$$

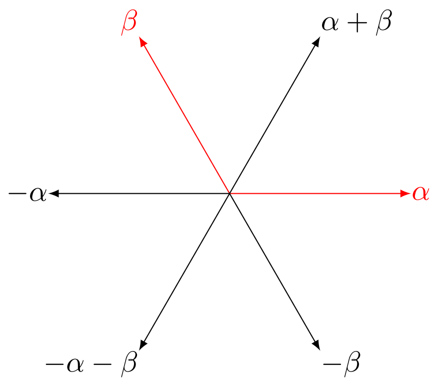
$$A_2 : \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \qquad G_2 : \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}.$$

B.3 Rank 2 Root System Diagrams

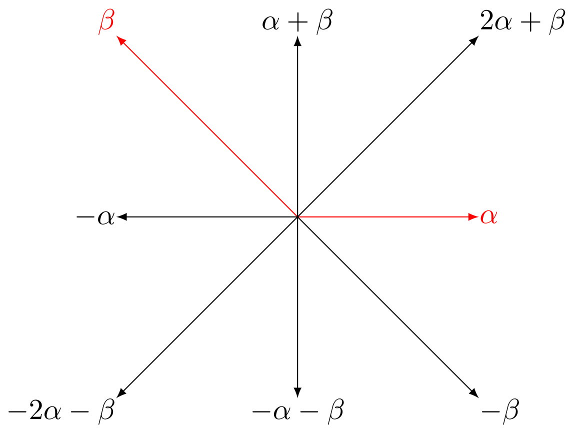
$A_1 \times A_1$:



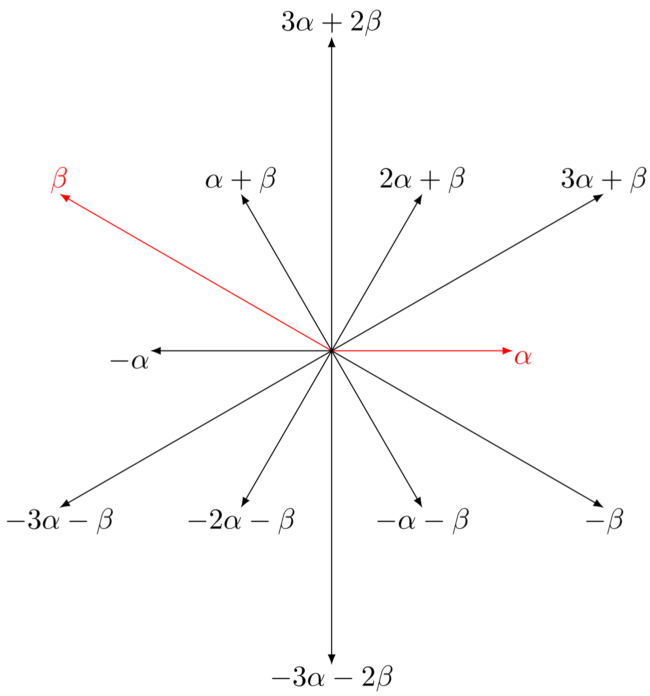
A_2 :



B_2 :


β
$\alpha + \beta$
$2\alpha + \beta$
$-\alpha$
α
$-2\alpha - \beta$
$-\alpha - \beta$
$-\beta$

G_2 :


$3\alpha + 2\beta$
β
$\alpha + \beta$
$2\alpha + \beta$
$3\alpha + \beta$
$-\alpha$
α
$-3\alpha - \beta$
$-2\alpha - \beta$
$-\alpha - \beta$
$-\beta$
$-3\alpha - 2\beta$

Bibliography

[1] P. Slodowy. Two notes on a finiteness problem in the representation theory of finite groups. *Algebraic groups and Lie groups: a volume of papers in honour of the late RW Richardson*, page 331, 1997.

[2] B. Külshammer. Donovan's conjecture, crossed products and algebraic group actions. *Israel J. Math.*, 92(1–3):295–306, 1995.

[3] M.W. Liebeck and G.M. Seitz. *Reductive subgroups of exceptional algebraic groups*. American Mathematical Society, 1996. ISBN 0821804618.

[4] M.W. Liebeck and G.M. Seitz. *The maximal subgroups of positive dimension in exceptional algebraic groups*. American Mathematical Society, 2004. ISBN 0821834827.

[5] D.I. Stewart. G-complete reducibility and the exceptional algebraic groups. *Arxiv preprint arXiv:1011.4835*, 2010.

[6] D.I. Stewart. The reductive subgroups of G_2. *J. Group Theory*, 13:117–130, 2010.

[7] J.-P. Serre. Moursund lectures. *arXiv preprint math/0305257*, 1998.

[8] M. Bate, B. Martin, and G. Roehrle. A geometric approach to complete reducibility. *Inventiones Mathematicae*, 161(1):177–218, 2005.

[9] A. Weil. Remarks on the cohomology of groups. *Annals of Mathematics*, 80(1): 149–157, 1964.

[10] J.E. Humphreys. *Linear algebraic groups*. Springer, 1975.

[11] T.A. Springer. *Linear algebraic groups*. Modern Birkhäuser Classics. Birkhäuser Boston, 2008. ISBN 9780817648398.

[12] K. Brown. *Cohomology of groups*, volume 87 of *Graduate Texts in Mathematics*. Springer, 1982.

[13] G. Kemper. A characterization of linearly reductive groups by their invariants. *Transformation groups*, 5(1):85–92, 2000.

[14] RW Richardson. On orbits of algebraic groups and Lie groups. *Bulletin of the Australian Mathematical Society*, 25(01):1–28, 1982.

[15] Benjamin MS Martin. Restrictions of representations of a surface group to a pair of free subgroups. *Journal of Algebra*, 225(1):231–249, 2000.

[16] B. Martin. Reductive subgroups of reductive groups in nonzero characteristic. *Journal of Algebra*, 262(2):265–286, 2003.

[17] R.W. Carter. *Simple groups of Lie type.* John Wiley & Sons Inc, 1989.

[18] B. Martin. Nonabelian 1-cocycles. *Report UKC/IMS/04/24 http://www.math.auckland.ac.nz/∼bmartin/cocycles.pdf*, 2004.